Define $\delta_{\lambda\mu} := \begin{cases} 1 \text{ if } \lambda = \mu \\ 0 \text{ otherwise} \end{cases}$

Now define the Hall Inner product $\langle . \rangle : \Lambda_k \times \Lambda_k \to \mathbf{Q}$ with a basis $\{c_i\}_{i \in \mathbf{N}}$ (e.g. e, p, or s) as:

$$\langle \sum_i a_i c_i, \sum_j b_j c_j \rangle = \sum_i a_i \langle c_i, \sum_j b_j c_j \rangle = \sum_i a_i \langle c_i, b_i c_i \rangle = \sum_i a_i b_i \langle c_i, c_i \rangle$$

Ex

$$\begin{aligned}\langle e_3 - 2e_{21} + e_{111}, 3e_3 - 3e_{21} + e_{111} \rangle_e &= \langle e_3, 3e_3 - 3e_{21} + e_{111} \rangle_e + \langle -2e_{21}, 3e_3 - 3e_{21} + e_{111} \rangle_e \\ &\quad + \langle e_{111}, 3e_3 - 3e_{21} + e_{111} \rangle_e \\ &= \langle e_3, 3e_3 \rangle_e + \langle -2e_{21}, -3e_{21} \rangle_e + \langle e_{111}, e_{111} \rangle_e \\ &= 3\langle e_3, e_3 \rangle_e + 6\langle e_{21}, e_{21} \rangle_e + \langle e_{111}, e_{111} \rangle_e \\ &= 3 + 6 + 1 = 10\end{aligned}$$

Notice that we can scale basis (e.g. $\langle p_e, p_e \rangle = e$) For variables $\{x_i\}_{i \in \mathbf{N}}, \{y_i\}_{i \in \mathbf{N}}$ define $\Lambda_k(X, Y)$ as the set of power series with the following properties:

1) $\Lambda_k(X, Y)$ is symmetric over x or over y (e.g. $f(x_1, x_2) = f(x_2, x_1)$)

2) $\Lambda_k(X, Y)$ is homogeneous of degree k over x or over y (e.g. $f(cx_1, cx_2) = c^k f(x_1, x_2)$)

Prop. If $\{u_i\}, \{v_i\}$ are linearly independent sets in Λ_k then their product $\{u_i(X), v_i(Y)\}$ is linearly independent in $\Lambda_k(X, Y)$

Pf. $\sum_j \sum_i A_{ij} u_j(X) v_i(Y) = 0$

$\forall i \in \mathbb{Z} : \sum_j A_{ij} u_j = 0$ (Because $\{v_i\}$ is linearly independent)

$\forall i, j \in \mathbb{Z} : A_{ij} = 0$ (Because $\{u_i\}$ is linearly independent)

Therefore $\{u_i(X), v_i(Y)\}$ is linearly independent. □

Prop. Suppose $\{u_\lambda | \lambda \vdash k\}, \{u'_\lambda | \lambda \vdash k\}\{v_\lambda | \lambda \vdash k\}, \{u'_\lambda | \lambda \vdash k\}$ form a basis for Λ_k where $_\lambda, v_\mu \rangle := \delta_{\lambda\mu}$ and $\langle \langle u'_\lambda, v'_\lambda \rangle \rangle := \delta_{\lambda\mu}$.

Define a $p(k) \times p(k)$ matrices A,B such that

$$u_\lambda = \sum_{\mu \vdash k} A_{\lambda\mu} u'_\mu$$

$$v_\lambda = \sum_{\mu \vdash k} B_{\lambda\mu} v'_\mu$$

Then the following properties are equivalent

(1) $A^t = B^{-1}$

(2) $\langle . \rangle = \langle \langle . \rangle \rangle$

(3) $F_k(u, v) = F_k(u', v')$

Pf. (1) $\iff$ (3)

$$\begin{aligned}\langle \langle u_\lambda, v_\mu \rangle \rangle &= \langle \langle \sum_{\mu \vdash k} A_{\lambda\mu} u'_\mu, \sum_{\mu \vdash k} B_{\lambda\mu} v'_\mu \rangle \rangle \\ &= \sum_{\alpha \vdash k} \sum_{\beta \vdash k} A_{\lambda\alpha} B_{\mu\beta} \langle \langle u'_\alpha, v'_\beta \rangle \\ &= \sum_{\alpha \vdash k} A_{\lambda\alpha} B_{\mu\beta}\end{aligned}$$

Therefore $\forall \lambda, \mu \vdash k : \langle\langle u_\lambda, v_\mu \rangle\rangle = \delta_{\lambda\mu} = \langle u_\lambda, v_\mu \rangle\rangle \iff A^t = B^{-1}$
(1) $\iff$ (3)

$$\begin{aligned} F_k(u,v) &:= \sum_{\lambda \vdash k} u_\lambda(X) v_\lambda(Y) \\ &= \sum_{\lambda \vdash k} (\sum_{\mu \vdash k} A_{\lambda\mu} u'_\mu)(\sum_{\mu \vdash k} B_{\lambda\mu} v'_\mu) \\ &= \sum_{\alpha \vdash k} \sum_{\beta \vdash k} (\sum_{\lambda \vdash k} A_{\lambda\alpha} B_{\lambda\beta}) u'_\alpha(X) v'_\beta(Y) \end{aligned}$$

($\Rightarrow$) ($A^t = B^{-1}$) means

$$= \sum_{\alpha \vdash k} \sum_{\beta \vdash k} u'_\alpha v'_\beta(Y) = F_k(u', v')$$

($\Leftarrow$)

$$\sum_{\alpha \vdash k} \sum_{\beta \vdash k} (\sum_{\lambda \vdash k} A_{\lambda\alpha} B_{\lambda\beta}) u'_\alpha(X) v'_\beta(Y) = F_k(u,v) = F_k(u', v') = \sum_{\alpha \vdash k} \sum_{\beta \vdash k} u'_\alpha(X) v'_\beta(Y)$$

Which implies that $A^t = B^{-1}$ $\square$

Def. A generalize permutation of length n is defined as $\pi = \langle \begin{matrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{matrix} \rangle$ with the following properties:

(1) $a_1 \leq a_2 \leq \dots \leq a_n$

(2) $A_j = a_{j+1} \Rightarrow b_j \leq b_{j+1}$

We then define $topwt(\pi) = \prod_{i=1}^n x_{a_i}, botwt(\pi) = \prod_{i=1}^n y_{a_i}$ and S_n to be the set of all π of length n.

Ex $S_2 : \langle \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \rangle, \langle \begin{matrix} 1 & 1 \\ 1 & 2 \end{matrix} \rangle, \langle \begin{matrix} 1 & 1 \\ 2 & 2 \end{matrix} \rangle, \langle \begin{matrix} 1 & 2 \\ 1 & 1 \end{matrix} \rangle, \langle \begin{matrix} 1 & 2 \\ 1 & 2 \end{matrix} \rangle, \langle \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \rangle, \langle \begin{matrix} 1 & 2 \\ 2 & 2 \end{matrix} \rangle, \langle \begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix} \rangle, \langle \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix} \rangle, \langle \begin{matrix} 2 & 1 \\ 2 & 1 \end{matrix} \rangle, \langle \begin{matrix} 2 & 1 \\ 2 & 2 \end{matrix} \rangle, \langle \begin{matrix} 2 & 2 \\ 1 & 1 \end{matrix} \rangle, \langle \begin{matrix} 2 & 2 \\ 1 & 2 \end{matrix} \rangle, \langle \begin{matrix} 2 & 2 \\ 2 & 2 \end{matrix} \rangle$

Prop. $\sum_{\pi \in S} topwt(\pi) botwt(pi) = \prod_{j=1}^\infty \prod_{i=1}^\infty \frac{1}{1 - x_j y_k}$

Def. n_j := multiplicity of j in λ (e.g. 111, $N_1 = 3$)

$z_\lambda := \prod_{j=1}^\lambda j^{n_j} n_j!$

Prop. $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$ (Pf.) Use the previous proposition.

Prop. $\forall n \geq 0 : h_n = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}$

Ex n=3 (3) (21) (111) $h_3 = \frac{1}{6} + \frac{1}{3} + 1 = \frac{5}{2}$