# Hypergeometric Series

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**Recall:** A series with $\frac{c_{n+1}}{c_n} = r$ for constant $r$ is called a geometric series.

**Defn:** Hypergeometric series is series with $\frac{c_{n+1}}{c_n} = \frac{P(n)}{Q(n)}$ for polynomials $P, Q$.

From now on, we deal with series in $\mathbb{C}$.

Since $\mathbb{C}$ alg. closed, we can write

$$\frac{c_{n+1}}{c_n} = \frac{(n+a_1)\cdots(n+a_p)z}{(n+b_1)\cdots(n+b_q)(n+1)}, \qquad z \in \mathbb{C}$$

Iteration gives:

$$c_n = \left(\frac{c_n}{c_{n-1}}\right)\left(\frac{c_{n-1}}{c_{n-2}}\right)\cdots\left(\frac{c_1}{c_0}\right)c_0 = \frac{(a_1)_n(a_2)_n\cdots(a_p)_n z^n}{(b_1)_n(b_2)_n\cdots(b_q)_n\, n!}c_0$$

Where $(a)_n := a(a+1)\cdots(a+n-1)$, $(a)_0 := 1$.

$$\Rightarrow \sum_{n=0}^{\infty} c_n = c_0 \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}$$

**Definition:** $${}_pF_q\left({a_1,\dots,a_p \atop b_1,\dots,b_q}; z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n} \cdot \frac{z^n}{n!}$$

Viewing ${}_pF_q$ as power series in $z$ centred at $0$, we use ratio test:

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{n^p}{n^{q+1}}z\right| = \begin{cases} 0 & \text{if } p<q+1 \\ |z| & \text{if } p=q+1 \\ \infty & \text{if } q+1<p \end{cases}$$

$\Rightarrow$ ${}_pF_q$ converges (absolutely) for
$$\begin{array}{ll} z=0 & \text{if } q+1<p \\ |z|<1 & \text{if } q+1=p \\ z\in\mathbb{C} & \text{if } p<q+1 \end{array}$$

**Examples:**

① $$\ln(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^{n+1} = \sum_{n=0}^{\infty} \frac{(1)_n(1)_n}{(2)_n}\frac{(-1)^n z^{n+1}}{n!} = z\,{}_2F_1\left({1,1 \atop 2}; -z\right)$$

② $$\arctan z = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}z^{2n+1} = \sum_{n=0}^{\infty}\frac{(1/2)_n(1)_n}{(3/2)_n}\cdot\frac{(-1)^n z^{2n+1}}{n!} = z\cdot{}_2F_1\left({1/2,1 \atop 3/2}; -z^2\right)$$

(using fact $\frac{(1/2)_n}{(3/2)_n} = \frac{1}{2n+1}$)

$$_2F_1\left(\begin{matrix}a, b\\ b\end{matrix}; z\right) = {}_1F_0\left(\begin{matrix}a\\ -\end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = \sum_{n=0}^{\infty} \binom{-a}{n}(-z)^n = (1-z)^{-a}$$

$|z| < 1$

Also, $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = {}_0F_0(-; z) =$

Pivoting a bit;

Definition: (Gamma function) $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$ for $Re(z) > 0$,

$\Gamma(z)$ is unique analytic continuation to meromorphic fn. on $z \in \mathbb{C} \setminus \{\mathbb{Z}^-\}$.

Remark: Gamma is a "generalization" of the factorial.

Prop a): $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N} \setminus \{0\}$.

Pf: Base case: $\Gamma(1) = \int_0^\infty t^{(1-1)} e^{-t}\, dt = \int_0^\infty e^{-t} dt = -\frac{1}{\infty} - (-e^0)$
$= 1 = 0!$

Inductive step: $\Gamma(z+1) = \int_0^\infty t^z e^{-t}\, dt$

Integrating by parts: $\left(\begin{matrix} u = t^z & dv = e^{-t} dt \\ du = zt^{z-1} dt & v = -e^{-t}\end{matrix}\right)$

$$= \left[-t^z e^{-t}\right]_0^\infty + \int_0^\infty z t^{z-1} e^{-t} dt$$

$$= \lim_{t \to \infty} (-t^z e^{-t}) + 0^z e^{-0} + z \int_0^\infty t^{z-1} e^{-t}\, dt$$

$$= z \int_0^\infty t^{z-1} e^{-t}\, dt = z\Gamma(z).$$

By inductive hypothesis, $\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$ □

Definition: $B(z_1, z_2) = \int_0^1 t^{z_1 - 1}(1-t)^{z_2 - 1}\, dt$ for $Re(z_1), Re(z_2) > 0$.

Prop b): $B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}$

Pf: $\Gamma(z_1)\Gamma(z_2) = \int_0^\infty e^{-u} u^{z_1 - 1} du \int_0^\infty e^{-v} v^{z_2 - 1} dv$

$$= \int_0^\infty \int_0^\infty e^{-u-v} u^{z_1 - 1} v^{z_2 - 1} du\, dv$$

subbing in $u = st$, $v = s(1-t)$:

$$\Gamma(z_1)\Gamma(z_2) = \int_{s=0}^\infty \int_{t=0}^\infty e^{-s} (st)^{z_1 - 1} (s(1-t))^{z_2 - 1} s\, dt\, ds$$

$$= \int_0^\infty e^{-s} s^{z_1 + z_2 - 1} ds \cdot \int_0^1 t^{z_1 - 1} (1-t)^{z_2 - 1} dt$$

$$= \Gamma(z_1 + z_2) \cdot B(z_1, z_2)$$

**Thm 1:** For $\operatorname{Re} c > \operatorname{Re} b > 0$,

$${}_2F_1\left({a,b \atop c};z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt$$

for $z \in \mathbb{C}\setminus(1,\infty)$.

Pf: Suppose $|z|<1$.

We have $(1-zt)^{-a} = \sum_{n=0}^{\infty}\binom{-a}{n}(-zt)^n = \sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n t^n$

$$\Rightarrow \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt = \sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n\int_0^1 t^{n+b-1}(1-t)^{c-b-1}\,dt$$

$$= \sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n B(n+b,c-b)$$

$$= \sum_{n=0}^{\infty}\frac{(a_n)}{n!}z^n\frac{\Gamma(n+b)\Gamma(c-b)}{\Gamma(n+c)}$$

(*) Noting that $\Gamma(n+b) = (n+b-1)\Gamma(n+b-1) = (n+b)((n+b-1)\Gamma(n+b-2))$

$$\vdots$$

$$= (n+b-1)\cdots(b+1)b\,\Gamma(b) = (b)_n\Gamma(b)$$

$$\Rightarrow \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt$$

$$= \frac{\Gamma(c)}{\cancel{\Gamma(c-b)}}\cdot\sum_{n=0}^{\infty}\frac{(a_n)}{n!}z^n\frac{\Gamma(n+b)}{\Gamma(b)}\frac{\cancel{\Gamma(c-b)}}{\Gamma(n+c)} = \sum_{n=0}^{\infty}\frac{(a_n)}{n!}\frac{\Gamma(n+b)}{\Gamma(b)}\frac{\Gamma(c)}{\Gamma(n+c)}$$

$$= \sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n\frac{(b)_n}{(c)_n} = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{z^n}{n!}$$

$$= {}_2F_1\left({a,b \atop c};z\right).$$

Thus, the thm holds for open set $|z|<1$. Since the integral representation is analytic on $\mathbb{C}\setminus[1,\infty)$ and agrees with ${}_2F_1$ on open set $|z|<1$, by uniqueness of analytic continuation it must hold for $\mathbb{C}\setminus(1,\infty)$.

**Thm 2:** For $\operatorname{Re}(c-a-b)>0$; ${}_2F_1\left({a,b \atop c};1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$

$\operatorname{Re}(c-a-b)$ $\operatorname{Re}(c)>\operatorname{Re} b>0$

Pf: by thm 1, cty of ${}_2F_1$, ${}_2F_1\left({a,b \atop c},1\right) = \lim_{z\to 1}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt$

$z\to 1$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1-a}\,dt = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}B(b,c-a-b) \text{ by defn}$$

$$= \frac{\Gamma(c)}{\cancel{\Gamma(b)}\Gamma(c-b)}\,\frac{\cancel{\Gamma(b)}\Gamma(c-a-b)}{\Gamma(c-a)}$$

For open disk $\operatorname{Re} c > \operatorname{Re} b > 0$, $\operatorname{Re}(c-a-b) > 0$. $\square$

<u>Cor. 3</u>: $c \notin \mathbb{Z}^-$, $\quad {}_2F_1\left({-n, b \atop c}; 1\right) = \frac{(c-b)_n}{(c)_n} \quad n \in \mathbb{N}$.

<u>Pf</u>: Subbing $a := -n$ into thm 2, we obtain

$$ {}_2F_1\left({-n, b \atop c}; 1\right) = \frac{\Gamma(c)\Gamma(c-(-n)-b)}{\Gamma(c-(-n))\Gamma(c-b)} = \left(\frac{\Gamma(c)}{\Gamma(c+n)}\right)\left(\frac{\Gamma((c-b)+n)}{\Gamma(c-b)}\right) = \frac{(c-b)_n}{(c)_n} $$

$\square$

<u>Thm 4</u>: $\quad {}_2F_1\left({a, b \atop c}; z\right) = (1-z)^{-a}\, {}_2F_1\left({a, c-b \atop c}; \frac{z}{z-1}\right)$

<u>Pf</u>: $\quad {}_2F_1\left({a, b \atop c}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt$

Subbing $t = 1-s$: $= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} = -\int_1^0 (1-s)^{b-1} s^{c-b-1}(1-z+zs)^{-a}\, ds$

$$ = (1-z)^{-a} \int_0^1 s^{c-b-1}(1-s)^{b-1}\left(1-\frac{zs}{z-1}\right)^{-a} ds $$

$\square$

<u>Thm 5</u>: $\quad {}_2F_1\left({a, b \atop c}; z\right) = (1-z)^{c-a-b}\, {}_2F_1\left({c-a, c-b \atop c}; z\right)$

<u>Pf</u>: $\quad {}_2F_1\left({a, b \atop c}; z\right) = (1-z)^{-a}\, {}_2F_1\left({a, c-b \atop c}; \frac{z}{z-1}\right) \quad$ by thm 4

$$ = (1-z)^{-a}\left(1-\frac{z}{z-1}\right)^{b-c} {}_2F_1\left({c-b, c-a \atop c}; \frac{\frac{z}{z-1}}{\frac{z}{z-1}-1}\right) $$

$$ = (1-z)^{c-a-b}\, {}_2F_1\left({c-b, c-a \atop c}; z\right) $$

since $\quad (1-z)\left(1-\frac{z}{z-1}\right) = (1-z)\left(1+\frac{z}{1-z}\right) = 1$. $\square$