Robinson Schenstad Algorithm

Review: Young Tableau: An array t obtained by replacing dots of the Ferrers diagram of a partition λ with the numbers $1, \dots, n$ bijectively.

example: If $\lambda = (2,1) = \begin{matrix} \bullet & \bullet \\ \bullet & \end{matrix}$, then

$$t: \begin{matrix} 1 & 2 \\ 3 & \end{matrix}, \quad \begin{matrix} 2 & 1 \\ 3 & \end{matrix}, \quad \begin{matrix} 1 & 3 \\ 2 & \end{matrix}, \quad \begin{matrix} 3 & 1 \\ 2 & \end{matrix}, \quad \begin{matrix} 2 & 3 \\ 1 & \end{matrix}, \quad \begin{matrix} 3 & 2 \\ 1 & \end{matrix}$$

- A Young Tableau of shape λ is also called a λ-tableau and denoted by t^λ.
- Can also write $\text{sh}\, t = \lambda$
- There are $n!$ Young Tableaux for any shape $\lambda \vdash n$.

A Tableau is standard if the rows and columns are increasing sequences.

$$f^\lambda := \text{The number of standard Tableaux of shape } \lambda$$

example: $t = \begin{matrix} 1 & 2 & 3 \\ 4 & 6 & \\ 5 & & \end{matrix}$ is standard, $t = \begin{matrix} 1 & 2 & 3 \\ 5 & 4 & \\ 6 & & \end{matrix}$ is not standard.

RS Algorithm motivation: Another way to prove that

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!,$$ just using combinatorics.

We denote the bijection by

$$\underset{S_n}{\overset{}{\pi}} \xleftrightarrow{R\text{-}S} (P, Q) \qquad \pi \in S_n$$

P and Q are standard λ-tableaux.

Algorithm Outline: We construct a sequence of tableau pairs (Seq. 3.1)

$$(P_0, Q_0) = (\emptyset, \emptyset), (P_1, Q_1), \dots, (P_n, Q_n) = (P, Q), \text{ where}$$
$$\text{sh}\, P_k = \text{sh}\, Q_k \ \forall k.$$

- At each step, we insert x_k into P_k and we place k into Q_k (Assuming $\pi = \begin{matrix} 1, 2, \dots, k \\ x_1, x_2, \dots, x_n \end{matrix}$)

The Insertion Algorithm: Let P be a partial tableau (i.e., an array with distinct entries whose rows and columns increase)

Pseudocode for the row insertion algorithm is as follows (We are inserting x into P):

- RS1 Set $R :=$ the first row of P
- RS2 While x is less than some element of row R, do
  - RSa Let y be the smallest element of R greater than x and replace y by x in R $(R \leftarrow x)$
  - RSb Set $x := y$ and $R :=$ the next row down
- RS3 Now x is greater than every element of R, so place x at the end of row R and stop.

example: Suppose $P = \begin{array}{llll} 1 & 2 & 5 & 8 \\ 4 & 7 & & \\ 6 & & & \\ 9 & & & \end{array}$ and we wish to insert $x=3$ into P.

Path of insertion:

$$\begin{array}{llll} 1 & 2 & 5 & 8 \\ 4 & 7 & & \\ 6 & & & \\ 9 & & & \end{array} \leftarrow 3 \;\Big|\; \begin{array}{llll} 1 & 2 & 3 & 8 \\ 4 & 7 & & \\ 6 & & & \\ 9 & & & \end{array} \leftarrow 5 \;\Big|\; \begin{array}{llll} 1 & 2 & 3 & 8 \\ 4 & 5 & & \\ 6 & & & \\ 9 & & & \end{array} \leftarrow 7 \;\Big|\; \begin{array}{llll} 1 & 2 & 3 & 8 \\ 4 & 5 & & \\ 6 & 7 & & \\ 9 & & & \end{array}$$

Notation: $r_x(P) = P'$ is the resulting tableau after row inserting x into P.

- Notice that P' will still have increasing rows and columns

The Placement Algorithm: Suppose Q is a partial tableau of shape μ and that (i,j) is an outer corner of μ. Merely set $Q_{ij} := k$ to place k in Q at cell (i,j).

example: $Q = \begin{array}{lll} 1 & 2 & 5 \\ 4 & 7 & \\ 6 & & \\ 8 & & \end{array}$, $Q' = \begin{array}{lll} 1 & 2 & 5 \\ 4 & 7 & 9 \\ 6 & & \\ 8 & & \end{array}$ (after placing $k=9$ at cell $(i,j) = (2,3)$)

Building Sequence 3.1: Start with our permutation $\pi = \begin{matrix} 1 & 2 & \dots & n \\ x_1 & x_2 & \dots & x_n \end{matrix}$ and a pair (P_0, Q_0) of empty tableaux.

- Define (P_k, Q_k) by:
  $P_k = r_{x_k}(P_{k-1})$
  Q_k = place k into Q_{k-1} at the cell (i,j) where the insertion terminates.

example: $\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 3 & 6 & 5 & 1 & 7 \end{matrix}$

P_k:	$\varnothing$	4	2 4	2 3 4	2 3 6 4	2 3 5 4 6	1 3 5 2 6 4	1 3 5 7 2 6 4 = P
Q_k:	$\varnothing$	1	1 2	1 3 2	1 3 4 2	1 3 4 2 5	1 3 4 2 5 6	1 3 4 7 2 5 6 = Q

Thm 3.1.1 The map $\pi \xrightarrow{R\text{-}S} (P, Q)$ is a bijection between elements of S_n and pairs of standard tableaux of the same shape

Pf Create an inverse map by reversing the RS algorithm

"Inverse" Algorithm: Assume we have (P_k, Q_k). Our goal is to find x_k and (P_{k-1}, Q_{k-1}). We will refer to P_k as P in order to avoid subscript confusion

- Find the cell (i,j) containing k in Q_k. We will <u>delete</u> $P_{i,j}$.

SR1 Set $x := P_{i,j}$ and erase $P_{i,j}$
Set $R :=$ the $(i-1)$st row of P_k

SR2 <u>While</u> R is not the 0^{th} row of P_k, <u>do</u>

SRa Let y be the largest element of R smaller than x, and replace y by x in R.

SRb Set $x := y$ and $R :=$ the next row up.

SR3 Now x has been removed from the first row, so we know that $x_k = x$. (i.e., x_k is what must have been inserted when creating P_k from P_{k-1})

example (going backwards): Let $P_n = \begin{smallmatrix} 1&3&5&7 \\ 2&6 \\ 4 \end{smallmatrix}$ and $Q_n = \begin{smallmatrix} 1&3&4&7 \\ 2&5 \\ 6 \end{smallmatrix}$. We wish to find π.

	$\overset{7}{\uparrow}$	$\overset{1}{\uparrow}$	$\overset{5}{\uparrow}$	$\overset{6}{\uparrow}$	$\overset{3}{\uparrow}$	$\overset{2}{\uparrow}$	$\overset{4}{\uparrow}$
P:	$\begin{smallmatrix} 1&3&5&7 \\ 2&6 \\ 4 \end{smallmatrix}$	$\begin{smallmatrix} 1&3&5 \\ 2&6 \\ 4 \end{smallmatrix}$	$\begin{smallmatrix} 2&3&5 \\ 4&6 \end{smallmatrix}$	$\begin{smallmatrix} 2&3&6 \\ 4 \end{smallmatrix}$	$\begin{smallmatrix} 2&3 \\ 4 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}$	4
Q:	$\begin{smallmatrix} 1&3&4&7 \\ 2&5 \\ 6 \end{smallmatrix}$	$\begin{smallmatrix} 1&3&4 \\ 2&5 \\ 6 \end{smallmatrix}$	$\begin{smallmatrix} 1&3&4 \\ 2&5 \end{smallmatrix}$	$\begin{smallmatrix} 1&3&4 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 1&3 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$	1
$k=$	7,	6,	5,	4,	3,	2,	1

$\Rightarrow \pi = 4236517$

Column Insertion: Simply replace the word "row" with the word "column" everywhere in our pseudocode to get

$$c_x(P) = P'$$

Prop 3.2.2: For any partial tableau P and distinct elements $x, y \notin P$,

$$c_y r_x(P) = r_x c_y(P)$$ (i.e., column and row insertion commute!)

Proof by examples: Let m be the maximum element in $\{x, y\} \cup P$.

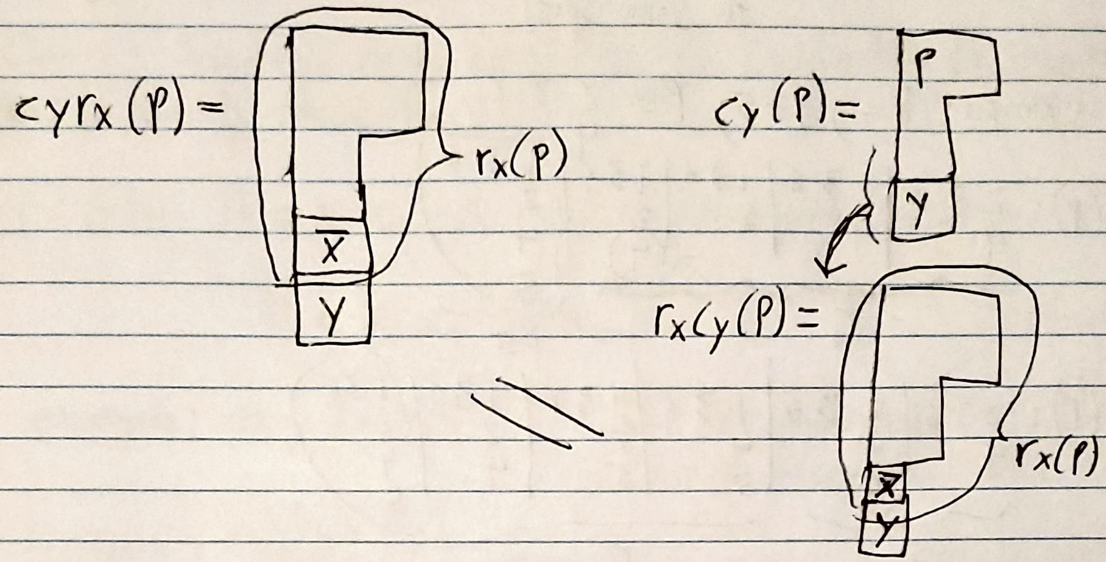
Notice: If we insert m into P, none of the previously existing elements of P would move.

Case 1 (real proof): Suppose $y = m$ (the case where $x = m$ is similar).

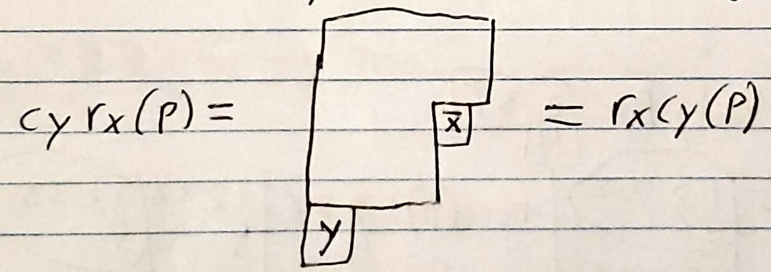
y is maximal $\Rightarrow$ c_y applied to P or to $r_x(P)$ places y at the end of the first column.

Let $\bar{x}$ = the last element to be "bumped" during the insertion $r_x(P)$.

If $\bar{x}$ lands in the first column:



If $\bar{x}$ lands in any other column: Same logic applies



Case 2: $m \in P$ (formal proof is by induction)

Induct on the number of entries in P.
Let $\bar{P}$ be P with the m erased.
$\Rightarrow cyrx(\bar{P}) \subset cyrx(P)$ and $rxcy(\bar{P}) \subset rxcy(P)$.
$cyrx(\bar{P}) = rxcy(\bar{P})$ by induction
$\Rightarrow cyrx(P)$ and $rxcy(P)$ agree everywhere except for maybe the location of m.
$\Rightarrow$ To show: m occupies the same position in both tableaux.

Let $\bar{x}$ be the last element displaced during $rx(\bar{P})$ into cell u.
Let $\bar{Y}$ __________________________ $cy(\bar{P})$ into cell V.

Subcase 2a: $u = v$ (i.e., $r_x(\overline{P})$ and $c_y(\overline{P})$ end by putting something into the same spot)

example: $x=1$, $y=5$, $P = \begin{array}{lll}2&3&6\\4\end{array}$

$$c_5 r_1(P):\ \underbrace{\begin{array}{lll}2&3&6\\4\end{array}\Bigg|\begin{array}{lll}1&3&6\\4\end{array}\Bigg|\begin{array}{lll}1&3&6\\2\end{array}\Bigg|\begin{array}{lll}1&3&6\\2\\4\end{array}}_{r_1}\Bigg|\underbrace{\begin{array}{lll}1&3&6\\2\\4\\5\end{array}}_{c_5}$$

$u = v = (3,1)$

$$r_1 c_5(P):\ \underbrace{\begin{array}{lll}2&3&6\\4\end{array}\Bigg|\begin{array}{lll}2&3&6\\4\\5\end{array}}_{c_5}\Bigg|\underbrace{\begin{array}{lll}1&3&6\\4\\5\end{array}\Bigg|\begin{array}{lll}1&3&6\\2\\5\end{array}\Bigg|\begin{array}{lll}1&3&6\\2\\4\end{array}\Bigg|\begin{array}{lll}1&3&6\\2\\4\\5\end{array}}_{r_1}$$

Subcase 2b: $u \neq v$

example: $x=5$, $y=1$, $P = \boxed{\begin{array}{lll}2&3&6\\4\end{array}}$

$$c_1 r_5(P):\ \underbrace{\begin{array}{lll}2&3&6\\4\end{array}\Bigg|\begin{array}{lll}2&3&5\\4&6\end{array}}_{r_5}\Bigg|\underbrace{\begin{array}{llll}1&2&3&5\\4&6\end{array}}_{c_1}$$

$$r_5 c_1(P):\ \begin{array}{lll}2&3&6\\4\end{array}\Bigg|\begin{array}{llll}1&2&3&6\\4\end{array}\Bigg|\begin{array}{llll}1&2&3&5\\4&6\end{array}$$

Thm 3.2.3: If $P(\pi) = P$, then $P(\pi^r) = P^t$ (t denotes transposition)

Pf

$$\begin{aligned} P(\pi^r) &= r_{x_1}\cdots r_{x_{n-1}} r_{x_n}(\emptyset) && \text{(by definition)}\\ &= r_{x_1}\cdots r_{x_{n-1}} c_{x_n}(\emptyset) && \text{(initial table is empty)}\\ &= c_{x_n} r_{x_1}\cdots r_{x_{n-1}}(\emptyset) && \text{(by commutativity)}\\ &\ \ \vdots && \text{(induction)}\\ &= c_{x_n} c_{x_{n-1}}\cdots c_{x_1}(\emptyset)\\ &= P^t \end{aligned}$$

$\square$

Increasing and Decreasing Subsequences

- One of Schensted's main motivations for constructing the insertion algorithm was to study these for a given π.

Def: Given $\pi = x_1 x_2 \cdots x_n \in S_n$, an increasing subsequence of π is

$$x_{i_1} < x_{i_2} < \ldots < x_{i_k}$$

example: $\pi = 4236517$. Increasing subsequence: 2357 ($k=4$)
Decreasing subsequence: 431 ($k=3$)

$$P(\pi) = \begin{array}{cccc} 1 & 3 & 5 & 7 \\ 2 & 6 & & \\ 4 & & & \end{array}$$

$\leftarrow$ Not a coincidence that the lengths match up!

Thm 3.3.2: for any $\pi \in S_n$, len(longest incr. sub.) = len(first row of $P(\pi)$), len(longest decr. sub.) = len(first column of $P(\pi)$)

Pf: first, notice that the second half follows if we prove the first half (thanks to thm 3.2.3)

Let P_{k-1} be the tableau formed after $k-1$ insertions of the RS alg.
Theorem follows from the lemma below.

Lemma 3.3.3: If $\pi = x_1 x_2 \cdots x_n$ and x_k enters P_{k-1} in column j, then the longest increasing subsequence of π ending in x_k has length j.

Pf Induct on k. Suppose statement holds for all values up to $k-1$.

First, we show that $\exists$ an increasing subsequence of length j ending in x_k.
Let y be the element of P_{k-1} in cell $(1, j-1)$.
$\Rightarrow y < x_k$ (since x_k enters in column j)
Inductive hypothesis $\Rightarrow \exists$ an increasing subsequence σ of length $j-1$ ending in y.
$\Rightarrow \sigma x_k$ is an increasing subsequence of length j.

All that remains is to show that there cannot be a longer increasing subsequence ending in x_k.

Suppose that such a sequence exists. Let x_i be the element preceding x_k in that subsequence.

Inductive hypothesis $\Rightarrow$ x_i first enters in some column to the right of column j.
If y is the element in cell $(1,j)$ of P_i, then

$$y \leq x_i < x_k.$$

Fact (from lemma 3.2.1 of textbook): Entries in a given position of a tableau never increase with subsequent insertions!

$\Rightarrow$ The element in cell $(1,j)$ of $P_{k-1} < x_k$
$\Rightarrow$ Contradiction! (x_k never would have entered column j, since it only replaces something that is larger)

$\square$