Schur Polynomials

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Spring 2023

1 Schur Functions and Semistandard Tableaux

First, we will add some new definitions to Ferrers diagrams so that we can state new propositions. Recall the convention that, for any filling T of a Ferrers diagram with positive integers, x^T denotes the monomial given by $\prod_{j \in J} x_j$. For the above example, the associated monomial is $x_1^2 x_2^2 x_3 x_4 x_6^2$.

Definition 1.1. Define *StrictRow(λ, A)* as the set of fillings of a Ferrers diagram of λ with entries in A in which each row is strictly increasing from left to right.

Proposition 1.2. For any partition λ, integer $n \geq 1$, we have the following two results:

$$e_\lambda(X_n) = \sum_{T \in StrictRow(\lambda, [n])} x^T \tag{1}$$

$$e_\lambda = \sum_{T \in StrictRow(\lambda, \mathbb{P})} x^T \tag{2}$$

Proof. We have that $e_\lambda(X_n) = \prod_{j=1}^{l(\lambda)} e_{\lambda_j}(X_n)$. Define t_j as $e_{\lambda_j}(X_n)$.
For each product t_j, we can construct a filling of the Ferrers diagram of λ by placing the subscript of each variable in t_j in increasing order in the j-th row of the diagram, $1 \leq j \leq l(\lambda)$. Likewise, for each filling of the Ferrers diagram, we can reconstruct t_j for $1 \leq j \leq l(\lambda)$ as the product $\prod_k x_k$, for all k that appear in the j-th row. Consequently, there is a bijection between $e_\lambda(X_n)$, the fillings of the diagram. Since the image of a filling T is x^T, (1) follows. $\square$

The proof for (2) is similar.

Example 1.3. Find all fillings T of the Ferrers diagram of (3^2, 1), associated with the term $x_1^2 x_2 x_4^3 x_6$ in e_{331}.
$e_{331} = e_3 e_3 e_1$, so we need to write $x_1^2 x_2 x_4^3 x_6$ as a product of 3 factors of degrees 3, 3, and 1, respectively. x_4 has degree 3, so it is in each factor, and x_1 has

degree 2, so it is in two of the factors, etc., so $x_1^2x_2x_4^3x_6 = (x_1x_2x_4)(x_1x_4x_6)x_4 = (x_1x_4x_6)(x_1x_2x_4)x_4$. This gives the two associated Ferrers diagrams:

$$\begin{array}{|c|c|c|}\hline 4 \\ \cline{1-1} 1 & 2 & 4 \\ \hline 1 & 4 & 6 \\ \hline\end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 4 \\ \cline{1-1} 1 & 4 & 6 \\ \hline 1 & 2 & 4 \\ \hline\end{array}$$

Definition 1.4. Define *WeakRow(λ, A)* as the set of fillings of a Ferrers diagram of λ with entries in A in which each row is weakly increasing from left to right.

Proposition 1.5. For any partition λ, integer $n \geq 1$, we have the following results:

$$h_\lambda(X_n) = \sum_{T \in WeakRow(\lambda,[n])} x^T \tag{3}$$

$$h_\lambda = \sum_{T \in WeakRow(\lambda,\mathbb{P})} x^T \tag{4}$$

The proofs for (3) and (4) are similar to the proof for (1).

Definition 1.6. For any λ, a *semistandard tableaux* of shape λ is a filling of a Ferrers diagram of λ with entries strictly increasing in columns (column-strict), and weakly increasing in rows (row-non-decreasing). $sh(T)$ denotes the shape of the SST, $SST(\lambda; n)$ denotes the set of all SSTs of shape λ with entries in $[n]$, and $SST(\lambda)$ is the set of all SSTs of shape λ with entries in $\mathbb{P}$.

Example 1.7. Find all semistandard tableaux of shape (2^2,1) with entries in $[3]$.
Since we know the columns are strictly increasing, and the rows are weakly increasing, it follows that we get:

$$\begin{array}{|c|c|}\hline 3 \\ \cline{1-1} 2 & 2 \\ \hline 1 & 1 \\ \hline\end{array} \qquad\qquad \begin{array}{|c|c|}\hline 3 \\ \cline{1-1} 2 & 3 \\ \hline 1 & 1 \\ \hline\end{array} \qquad\qquad \begin{array}{|c|c|}\hline 3 \\ \cline{1-1} 2 & 3 \\ \hline 1 & 2 \\ \hline\end{array}$$

Using semistandard tableaux, we can construct new polynomials and power series, which will hopefully be symmetric.

Definition 1.8. *Schur polynomial*: $s_\lambda(X_n) = \sum\limits_{T \in SST(\lambda;n)} x^T$

Definition 1.9. *Schur function*: $s_\lambda = \sum\limits_{T \in SST(\lambda)} x^T$

Example 1.10. Find $s_{1^k}(X_n), s_{1^k}$
$s_{1^k}(X_n)$ is a sum over all fillings of a single column with distinct integers in $[n]$, where the column is column-strict. For $n < k$, no such fillings will exist, $\therefore s_{1^k}(X_n) = 0$.
For $n \geq k$, there will be a unique filling of the Ferrers diagram for each subset of $[n]$ that has size k. Denoting the set of subsets with size k as J, we have the monomial $\prod_{j \in J} x_j, \therefore s_{1^k}(X_n) = e_k(X_n)$. Similarly, $s_{1^k} = e_k$

Example 1.11. Find $s_k(X_n), s_k$
$s_k(X_n)$ is a sum over all possible fillings of a single row with integers in [n] that are weakly increasing from left to right. For each submultiset, there will exist a unique filling, where the monomial corresponding to each submultiset J is $\prod_{j \in J} x_j$, meaning $s_k(X_n) = h_k(X_n)$, and similarly $s_k = h_k$.

Example 1.12. Express $s_{21}(X_1), s_{21}(X_2)$, and $s_{21}(X_3)$ as linear combinations of symmetric polynomials.
s_{21} is a sum over all SST such that $sh(T) = (2,1)$, with entries in $[1]$. The left-most column must have 2 distinct integers, but we only have one, so $s_{21}(X_1) = 0$
$s_{21}(X_2)$ has two possibilities:

$$\begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 1 \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 2 \\ \hline \end{array}$$

From the above, we get that $s_{21}(X_2) = x_1^2 + x_2 + x_1 x_2^2 = m_{21}(X_2)$.
The steps to find $s_{21}(X_3)$ are omitted for the sake of brevity. One will find that $s_{21}(X_3) = m_{21}(X_3) + 2m_{111}(X_3)$

Definition 1.13. For any semistandard tableaux T, the content of T is the sequence $\{\mu_j\}_{j=1}^{\inf}$, where μ_j is the number of j's in T for each j. When $\mu_j = 0$ for $j > n$, we write $\{\mu_j\}_{j=1}^{n}$

Proposition 1.14. Suppose $\lambda \vdash k, \lambda \geq 1$. Then, $s_\lambda(X_n) \in \Lambda(X_n)$, and $s_\lambda \in \Lambda_k$.

Definition 1.15. For any partitions λ, μ, the *Kostka numbers* $K_{\lambda,\mu}$ denote the number of semistandard tableaux with shape λ and content μ.

Remark 1.16. For any partition λ, $s_\lambda = \displaystyle\sum_{\mu \vdash |\lambda|} K_{\lambda,\mu} m_\mu$

Proposition 1.17. For any λ, μ such that $|\lambda| = |\mu|$, $n \in \mathbb{N}$, let $K_{\lambda,\mu,n}$ be the number of semistandard tableux with shape λ, content μ, and entries in $[n]$. If $n \geq \lambda, K_{\lambda,\mu,n} = K_{\lambda,\mu}$, i.e. $K_{\lambda,\mu,n}$ is independent of n.

Proof. μ is a partition, $n \geq |\lambda| = |\mu| \Rightarrow \mu_j = 0$ for $j \geq n$, which means that any semistandard tableaux with shape λ and content μ cannot have entries larger than n, so $K_{\lambda,\mu,n}$ and $K_{\lambda,\mu}$ count the same semistandard tableux, which gives the conclusion that $K_{\lambda,\mu,n} = K_{\lambda,\mu}$. □

Example 1.18. We now want to show that Schur functions of degree k are a basis for Λ_k, which we can do by examining the coefficients when we write s_λ as a linear combination of monomial symmetric functions. We can do this by looking at matrices of Kostka numbers.
$K_{1,1} = \begin{bmatrix} 1 \end{bmatrix}$
$K_{2,2} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$K_{3,3} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$K_{4,4} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Proposition 1.19. $\forall k \geq 0, \lambda \vdash k, \mu \vdash k$

1. $\mu >_{\text{lex}} \lambda \Rightarrow K_{\lambda,\mu} = 0$
2. $K_{\lambda,\lambda} = 1$

Proof. $\mu >_{\text{lex}} \lambda \Rightarrow \exists m$ such that $\mu_m > \lambda_m, \mu_j = \lambda_j$ for $1 \leq j \leq m-1$
No semistandard tableaux can have a 1 in its 2nd row or higher, a 2 in its 3rd row or higher, a j in the $j+1$-th row or higher, etc. Consequently, a filling T of λ with content μ can be semistandard tableaux if and only if it has j's in every entry of the j-th row for $1 \leq j \leq m-1$. Since $\mu_m > \lambda_m$, then by the pigeonhole principle some column of T has two m's, therefore it is not a semistandard tableaux. This proves 1, and the proof for 2 is similar. □

Proposition 1.20. $\forall k \geq 0, \{s_\lambda | \lambda \vdash k\}$ is a basis for Λ_k

Proof. Similar to the proof that $\{e_\lambda | \lambda \vdash k\}$ is a basis for Λ_k □

2 Schur Polynomials as Ratios of Determinants

Definition 2.1. For integer $n \geq 1$, a polynomial $f(X_n)$ is *alternating* whenever $\pi(f) = \text{sgn}(\pi) f \quad \forall \pi \in S_n$

Example 2.2. Find the alternating polynomial $f(X_3)$ with the fewest terms, containing $x_1^3 x_2^2 x_3$
$\pi(\,) \Rightarrow \pi(x_1^3 x_2^2 x_3) = x_1^3 x_2^2 x_3$, $\text{sgn}(\pi) = 1$
$\pi(1\ 2) \Rightarrow \pi(x_1^3 x_2^2 x_3) = x_1^3 x_2 x_3^2$, $\text{sgn}(\pi) = -1$
$\pi(1\ 2) \Rightarrow \pi(x_1^3 x_2^2 x_3) = x_1^2 x_2^3 x_3$, $\text{sgn}(\pi) = -1$
$\pi(2\ 3) \Rightarrow \pi(x_1^3 x_2^2 x_3) = x_1^3 x_2^3 x_3$, $\text{sgn}(\pi) = -1$
$\pi(1\ 3) \Rightarrow \pi(x_1^3 x_2^2 x_3) = x_1 x_2^2 x_3^3$, $\text{sgn}(\pi) = -1$
$\pi(1\ 2\ 3) \Rightarrow \pi(x_1^3 x_2^2 x_3) = x_1 x_2^3 x_3^2$
$\pi(1\ 3\ 2) \Rightarrow \pi(x_1^3 x_2^2 x_3) = x_1^2 x_2 x_3^3$
$\Rightarrow f(X_n) x_1^3 x_2^2 x_3 - x_1^3 x_2 x_3^2 - x_1^2 x_2^3 x_3 - x_1 x_2^2 x_3^3 + x_1 x_2^3 x_3^2 + x_1 x_2^3 x_3^2$

Example 2.3. Show that there does not exist an alternating polynomial with $x_1 x_2 asaterm$
If $x_1 x_2$ is a term, then for $\pi = (1\ 2)$, $\text{sgn}(\pi) = -1$, and $-x_1 x_2$ must also be a term, which is a contradiction.

Proposition 2.4. If $\mu_1, \cdots, \mu_n$ are nonnegative integers such that $u_j \neq u_l$ for $j \neq l$, and $f(X_n)$ is an alternating polynomial for X_n. then the coefficient of $x_1^{\mu_1} \cdots x_n^{\mu_n}$ is 0

Proof. If $ax_1^{\mu_1} \cdots x_n^{\mu_n} \in f(X_n)$, then sgn$((j\ l))(ax_1^{\mu_1} \cdots x_n^{\mu_n}) = -ax_1^{\mu_1} \cdots x_n^{\mu_n} \Rightarrow a = 0$. □

Remark 2.5. The above proposition means that we can construct nonzero alternating polynomials from monomials with distinct exponents, which will allow us to show that the alternating polynomial can be expressed as a determinant.

Proposition 2.6. If μ is a sequence such that $\mu_1 > \mu_2 > \cdots > \mu_n \geq 0$, then $a_\mu(X_n) = \sum_{\pi \in S_n} sgn(\pi) x_{\pi(1)}^{\mu_1} x_{\pi(2)}^{\mu_2} \cdots x_{\pi(n)}^{\mu_n}$, then $a_\mu(X_n)$ is an alternating polynomial in $x_1, x_2, \cdots, x_n$. Further, $a_\mu(X_n)$ is homogeneous of degree $\mu_1 + \cdots + \mu_n$, has $n!$ terms, and $a_\mu(X_n) = det(x_l^{\mu_j})_{1 \leq j, l \leq n}$

Proof. We know that for $\sigma \in S_n, \sigma(a_\mu) = \sum_{\pi \in S_n} \text{sgn}(\pi) x_{\sigma\pi(1)}^{\mu_1} \cdots x_{\sigma\pi(n)}^{\mu_n}$. If $\tau = \sigma\pi$, then $\pi = \sigma^{-1}\tau$, so as π ranges over S_n, so does $\tau, \therefore \sigma(a_\mu) = \sum_{\tau \in S_n} \text{sgn}(\sigma)\text{sgn}(\tau) x_{\tau(1)}^{\mu_1} \cdots x_{\tau(n)}^{\mu_n} = \sum_{\pi \in S_n} \text{sgn}(\sigma^{-1}\tau) x_{\tau(1)}^{\mu_1} \cdots x_{\tau(n)}^{\mu_n}$. Consequently, each term is of degree $\mu_1 + \cdots + \mu_n$, consequently it is homogeneous of degree $\mu_1 + \cdots + \mu_n$. Further, $\mu_1, \cdots, \mu_n$ are distinct, so all the terms of $a_n(X_n)$ are distinct, so there are $n!$ terms. Using the fact that $det(A) = \sum_{\pi \in S} \text{sgn}(\pi) \prod_{j=1}^{n} A_{j\pi}(j)$, the statement follows. □

Remark 2.7. It is more convenient to use partitions to index rather than integers. Using an inductive proof, we find that $\mu_{n-j} \geq j$ for $0 \leq j \leq n-1$, denote this as δ_n. Define λ by $\lambda_j = \mu_j - \delta_n(j), 1 \leq j \leq n. \lambda$ is a partition with at most n parts, so the map from $\mu \to \lambda$ is bijective, so we can view $a_\mu(X_n)$ as $a_{\lambda+\delta_n}(X_n)$

Proposition 2.8. $\forall n \geq 1, a_{\delta_n}(X_n) = \prod_{1 \leq j < l \leq n} (x_j - x_l)$. For all $n \geq 1$, and partitions λ with at most n parts, there exists a symmetric polynomial $g(X_n)$ such that $a_{\lambda+\delta_n}(X_n) = g(X_n) a_{\delta_n}(X_n)$.

Proof. First, we show that $(x_i - x_j)|P(X_n), (x_l - x_k|P(X_n) \Rightarrow (x_i - x_j)(x_l - x_k)|p(x_n)$, which is true since $(x_i - x_j)$ are prime in $\mathbb{C}[x_1, \cdots, x_n]$

$$a_{\delta_n}(X_n) = \begin{bmatrix} x_1^{n-1} & x_2^{n-1} & \cdots \\ x_1^{n-2} & \ddots & \\ \vdots & & \end{bmatrix}$$

If we set $x_1 = x_2$, then the first two columns are equivalent, meaning the determinant is 0. The same is true for $x_1 = x_i, 2 \leq i \leq n = 0$ We find that, repeating this $\binom{n}{2}$ times gives us that the determinant is a polynomial with degree $\binom{n}{2}$.

Similarly, $\det(x_l^{\mu_j})_{1\leq j\leq n} = \begin{vmatrix} x_1^{\mu_1} & x^{\mu_1} & \dots \\ x_1^{\mu_2} & x_2^{\mu_2} & \dots \\ \vdots & \vdots & \ddots \end{vmatrix}$

The above determinant gives us that $(x_1 - x_2)(x_1 - x_3)\dots(x_{n-1} - x_n)$, which divides a_{δ_n} by repeated use of the lemma. But this also has degree polynomial also has degree $\binom{n}{2}$, which gives that $a_{\delta_n}(x) = kQ(x), k \in \mathbb{C}$, and by checking we get that $k = 1$.

For the second part of the proposition, the same reasoning as above gives us that $a_{\delta_n}(X_n) = (x_1 - x_2)\dots(x_{n-1} - x_n|a_{\lambda+\delta_n}(x))$, and deg $a_{\lambda+\delta n}(x) > a_{\delta_n}(X)$ $\square$