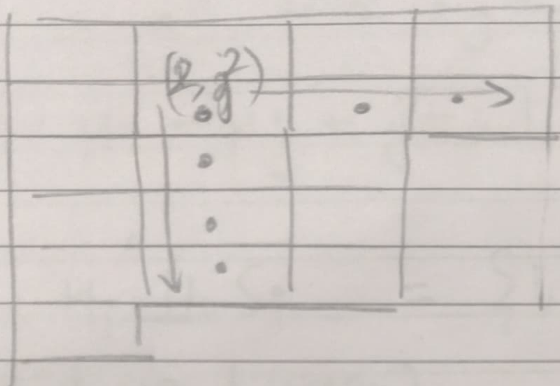
What are hooks?

if $v = (i, j)$ is node in diagram tableaux



$h_{(2,2)} = 6$

Hook length of $(i,j) \to h_{i,j}$ or $|h_{i,j}|$

HK-Content

$$\# |SSYT_\mu [N]| = \prod_{(i,j)\in\mu} \frac{N+j-i}{h(i,j)} = F_\mu (N)$$

$\mu = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ $N = 3$

$\begin{array}{cc} 1 & 1 \\ 2 & \end{array}$ $\begin{array}{cc} 1 & 1 \\ 3 & \end{array}$ $\begin{array}{cc} 1 & 2 \\ 2 & \end{array}$ $\begin{array}{cc} 1 & 2 \\ 3 & \end{array}$

$\begin{array}{cc} 1 & 3 \\ 2 & \end{array}$ $\begin{array}{cc} 1 & 3 \\ 3 & \end{array}$ $\begin{array}{cc} 2 & 2 \\ 3 & \end{array}$ $\begin{array}{cc} 2 & 3 \\ 3 & \end{array}$

8 total. Contents $(j-i)$ Hook-len.

$\begin{array}{cc} 0 & 1 \\ -1 & \end{array}$ $\begin{array}{cc} 3 & 1 \\ 1 & \end{array}$

Camlin Page
Date / /

part of HK-Content proof

<u>Def</u>: $h_\mu(i,a)$ = hook leng in row i, col a.

<u>Def</u>. $H_i = \{ h_\mu(i,a) \mid 1 \le a \le \mu_i \}$ μ_i = # boxes in row i

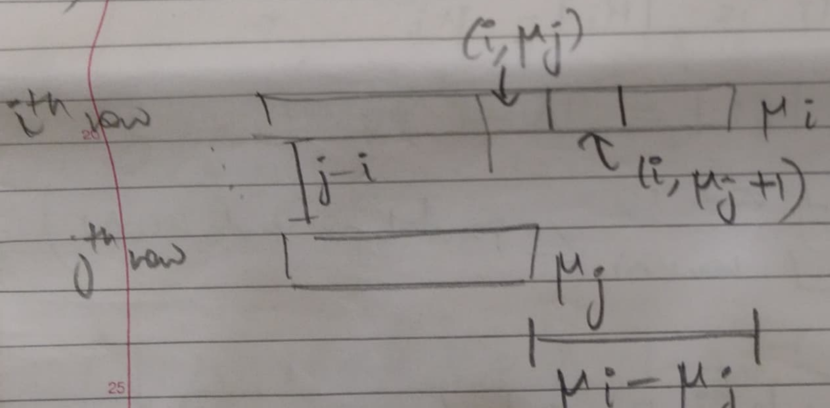
N = # parts of μ.

So, H_i is all hook leng the across row i

<u>Def</u>. $S_i = \{ \mu_i - \mu_j + j - i \mid j > i \}$

<u>Claim</u> $H_i \amalg S_i = \{1, 2, \ldots, \mu_i + N - i\}$ (disjoint union)

First, Show that two sets H_i & S_i are disjoint.



So, $\mu_i - \mu_j + j - i + 1 \le h_\mu(i, \mu_j)$

$h_\mu(i, \mu_j + 1) \ge \mu_i - \mu_j + j - i - 1$

So, $\mu_i - \mu_j + (j-i)$ cannot be a hook length in these 2 boxes

Claim: $h_\mu(i,a)$ strictly decreases as a increases

Since as we move to right, the horizontal hook length strictly dec whereas vertical hook length may decrease but cannot increase.

So, $(\mu_i - \mu_j + j - i)$ cannot be a hook-length in that row i

So, H_i disjoint from S_i.

Cardinality of H_i is $|H_i| = \mu_i$

Cardity of S_i $|S_i| = (N-i)$

(Since row i

row N ← partition ends row N)

& max in H_i is $h(i,0)$: μ_i

N

So, $h(i,0) = \mu_i + N - i$

So, $H_i \amalg S_i = \{2, \dots, (\mu_i + N - i)\}$

Content Pf: Second Part

(Take this as given)

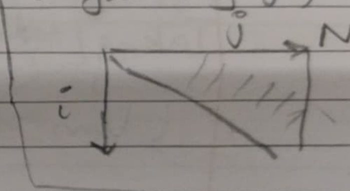
$$s_\mu(1,q,\dots,q^{n-1}) = \prod_{i<j} \frac{1-q^{\mu_i-\mu_j+j-i}}{1-q^{j-i}}$$

Remnant of S_i

① $$\prod_{i<j} 1-q^{\overbrace{\mu_i-\mu_j+j-i}} = \prod_{i=1}^{N} \frac{\prod_{j=1}^{\mu_i+N-i}(1-q^j)}{\prod_{a=1}^{\mu_i} 1-q^{h(i,a)}}$$

~~avg~~ $S_i \sqcup H_i = \{1,2,\dots,\mu_i+N-i\}$

Useful img fig:



② $$\prod_{i<j\le N} 1-q^{j-i} = \prod_{i=1}^{N} \prod_{j=1}^{N-i} 1-q^j$$

(Since $j>i$, $(j-i)$ ranges from 1 to $(N-i)$)

$$\prod_{i<j} 1-q^{j-i} = \prod_{j=2}^{N} 1-q^{j-1} \prod_{2<i<j} 1-q^{j-i}$$

①/② $\Rightarrow$ $$s_\mu(1,q,\dots,q^{n-1}) = \prod_{i=1}^{N} \prod_{j=N-i+1}^{\mu_i+N-i} (1-q^{i})$$
$$\prod_{a=1}^{\mu_i} 1-q^{h(i,a)}$$

Start from $j=N-i+1$ because ~~divide~~ $j\le N-i$ gets cut out

$$\prod_{i=1}^{N} \frac{\prod_{j=1}^{\mu_i} (1-q^{N-i+j})}{\prod_{a=1}^{\mu_i} 1-q^{h(i,a)}} = \prod_{(i,j)\in\mu} \frac{1-q^{N+j-i}}{1-q^{h(i,j)}}$$

So, $$S_\mu(1,q,\dots,q^{N-1}) = \prod_{(i,j)\in\mu} \frac{1-q^{N+j-i}}{1-q^{h(i,j)}}$$

Rem: $S_\mu(\underbrace{1,\dots,1}_{N \text{ times}}) = F_\mu(N) = \# SSYT_\mu([N])$

Taking limit as $q \to 1$ using L'Hospital Rule

(lim of num & lim of den both 0)

$$F_\mu(N) = \prod_{(i,j)\in\mu} \frac{N+j-i}{h(i,j)}$$

HOOK-Content Formula

From formula, clear that a greater N

(more entries we can use) much greater no. of

$|SSYT([N])|$

Hook length Form Examples:

$$f^{\lambda} = \frac{|\lambda|!}{\prod_{(i,j)\in\mu} h(i,j)} = \left|\text{Std. Y. Tab}_{\lambda}(n)\right|$$

$$n = |\lambda|$$

$$\mu = \begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1}\end{array}$$

Hook-length:

$$\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

$$f^{\lambda} = \frac{4!}{4\times2\times1} = \frac{1\times2\times3\times4}{4\times2} = 3$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$$

With class: $\mu = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \\ \cline{1-1}\end{array}$

Hk-length: $\begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$

$$f^{\lambda} = \frac{5!}{4\cdot3\cdot2\cdot1} = 5$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$$

...f of Hook-Length:

We want calculate f^{μ} : $|\mathrm{StdYT}_{\mu}[N]|$

We have formula for $F_{\mu}(N) = |\mathrm{SSYT}_{\mu}([N])|$

Need to link f^{μ} with $F_{\mu}(N)$:-

Using Cauchy-Littlewood Formula (No proof)

$$\prod_{i,j\geq 1} \frac{1}{1-x_i y_j} = \sum_{\lambda} s_{\lambda}(\vec{x})\, s_{\lambda}(\vec{y})$$

Here use, $\vec{x} = x_1, x_2, \ldots$ & $y_j = 1/N$ $(1 \leq j \leq N)$

Get,

$$\sum_{\mu} s_{\mu}(\vec{x}) \frac{F_{\mu}(N)}{N^{|\mu|}} = \prod_{i=1}^{N} \left(1 - \frac{x_i}{N}\right)^{-N}$$

(~~Not~~ Since $S_{\mu}(\vec{y}) = S_{\mu}(1/N, \ldots 1/N)$ ← There are $F_{\mu}(N)$ terms ~~total~~ & each term is $\left(\frac{1}{N}\right)^{|\mu|}$

~~So~~ get LHS, ~~to~~

To get RHS, $\frac{1}{1-x_i y_j} = \frac{1}{\left(1-\frac{x_i}{N}\right)}$

Since all y_j are same get $\prod_{i=1}^{N} \left(1-\frac{x_i}{N}\right)^{-1} \times \cdots \times \left(1-\frac{x_i}{N}\right)^{-1}$

Taking Lt as $N \to \infty$, get

$$\sum_{\mu} s_\mu(\vec{x}) \lim_{N\to\infty} \frac{F_\mu(N)}{N^{|\mu|}} = e^{h_1(\vec{x})}$$

(Using $e^{x_i} = \lim_{N\to\infty}\left(1+\frac{x_i}{N}\right)^N$ from calc get)

So, $e^{x_1+x_2+\dots}$

Eq 4.10 from text book

$$(x_1+x_2+\dots)^n = h_1^n(x) = \sum_{\lambda\vdash n} f^\lambda s_\lambda(\vec{x})$$

$$\text{So, } \sum_{\mu} s_\mu(\vec{x}) \lim_{N\to\infty} \frac{F_\mu(N)}{N^{|\mu|}} = e^{h_1(\vec{x})}$$

(calculus)
$$= \sum_{n\geq 0} \frac{h_1^n(\vec{x})}{n!} \underset{(4.10)}{=} \sum_{n\geq 0}\sum_{\lambda\vdash n} \frac{f^\lambda s_\lambda(\vec{x})}{n!}$$

$$= \sum_{\lambda} \frac{f^\lambda s_\lambda(\vec{x})}{|\lambda|!}$$

since s_λ linearly independent

$$\boxed{\text{So, } \lim_{N\to\infty} \frac{F_\mu(N)}{N^{|\mu|}} = \frac{f^\mu}{|\mu|!}}$$

From Hk. Content, we know.

$$F_\mu(N) = \prod_{(i,j)\in\mu} \frac{N+j-i}{h(i,j)}$$

So,
$$\frac{f^\mu}{|\mu|!} = \lim_{N\to\infty} \prod_{(i,j)\in\mu} \frac{1+(j-i)/N}{h(i,j)}$$

$$\boxed{\frac{f^\mu}{|\mu|!} = \prod_{(i,j)\in\mu} \frac{1}{h(i,j)}}$$

Hook-length formula

Examples: