Monomial Symmetric Function

Sunday, January 30, 2022 11:18 AM

Def: Let λ be a partition. The monomial symmetric Function $m_\lambda(X_n) = k\, m'_\lambda(X_n)$ where

- $m'_\lambda(X_n) = \sum_{\pi \in S_n} \pi(x_1^{\lambda_1} \cdots x_{\ell(\lambda)}^{\lambda_{\ell(\lambda)}})$
- $k \in \mathbb{Q}$ s.t. $\langle x_1^{\lambda_1} \cdots x_{\ell(\lambda)}^{\lambda_{\ell(\lambda)}} \rangle m_\lambda(X_n) = 1$

Ex: $\lambda = (1,1)$, $n = 3$

$$m'_\lambda(X_3) = (x_1 x_2 + (12)x_1 x_2 + (13)x_1 x_2 + (123)x_1 x_2 + (132)x_1 x_2 + (23)x_1 x_2)$$
$$= x_1x_2 + x_1x_2 + x_2x_3 + x_2x_3 + x_1x_3 + x_1x_3$$
$$= 2(\underbrace{x_1x_2 + x_2x_3 + x_1x_3}_{m_\lambda(X_3)})$$
$$\Rightarrow k = \tfrac{1}{2}$$

Lemma 1: $m_\lambda(X_n)$ sym $\Leftrightarrow$ $m'_\lambda(X_n)$ sym

Prop 2: $m'_\lambda(X_n)$ is sym

Pf: Let $\sigma \in S_n$, $\ell = \ell(\lambda)$. Then

$$\sigma(m'_\lambda(X_n)) = \sum_{\pi \in S_n} \sigma(\pi(x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell}))$$
$$= \sum_{\pi \in S_n} (\sigma \circ \pi)(x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell}) \overset{\star}{=} \sum_{\pi' \in S_n} \pi'(x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell})$$

$\star$ $\{\sigma \circ \pi\}_{\pi \in S_n} \underset{\sigma^{-1}}{\overset{\sigma}{\rightleftarrows}} \{\pi'\}_{\pi' \in S_n}$ $\quad \sigma \circ \sigma^{-1} = id$, $\sigma^{-1} \circ \sigma = id$

$\{\pi'\}_{\pi' \in S_n} = \{\pi \circ \sigma\}_{\pi \in S_n}$

$$= m'_\lambda(X_n)$$

Prop 3: $\{m_\lambda(X_n)\}_{\lambda \vdash k}$ is a basis for $\Lambda_k(X_n)$

Pf: Span: Let $f \in \Lambda_k(X_n)$, let $c\, x_1^{e_1} \cdots x_n^{e_n}$ be some monomial appearing in f. b/c $f = \pi f$ $\forall \pi \Rightarrow$

$$\ldots + c\, x_1^{e_1} \cdots x_n^{e_n} + \ldots + d\, \pi(x_1^{e_1} \cdots x_n^{e_n}) = \ldots + c\,\pi(x_1^{e_1} \cdots x_n^{e_n}) + \ldots$$

Fact: $\{x_1^{k_1}\cdots x_n^{k_n}\}_{k_i \in \mathbb{Z}\ge 0}$ form a basis for $\mathbb{C}[x_1,\dots,x_n]$

$\Rightarrow c=d \quad \forall \pi$

$\Rightarrow c\sum_{\pi\in S_n}\pi(x_1^{e_1}\cdots x_n^{e_n})$ appears in f

$\| \quad$ by ☆. Choose

$c\sum_{\pi\in S_n}\pi(\sigma(x_1^{e_1}\cdots x_n^{e_n}))$ σ s.t. $\sigma(x_1^{e_1}\cdots x_n^{e_n})$ $= x_1^{\lambda_1}\cdots x_n^{\lambda_n}$, $\lambda_1\ge\lambda_2\ge\cdots$

$\|$

$c\, m_\lambda'(X_n)$

$\Rightarrow f - \text{"}c\,m_\lambda'(X_n)\text{"} \in \Lambda_k(X_n)$. Repeat/use induction

$\Rightarrow m_\lambda'(X_n)$ spans $\Lambda_k(X_n) \Leftrightarrow m_\lambda(X_n)$ spans $\Lambda_k(X_n)$

L.I. check ✓

Remark: The "$c=d$" part of pf $\Rightarrow$ If $f=\sum_{\lambda\vdash k} a_\lambda m_\lambda$, to find $a_\lambda \Leftrightarrow$ finding coefficent of $x_1^{\lambda_1}\cdots x_n^{\lambda_n}$ in $f =: \langle x_1^{\lambda_1}\cdots x_n^{\lambda_n}\rangle f$

Prop 4 $\{e_\lambda\},\{h_\lambda\},\{p_\lambda\},\{s_\lambda\}$ $\lambda\vdash k$ are bases for $\Lambda_k(X_n)$

Pf: Except for $\{h_\lambda\}$ in other cases we saw that

$$?_\lambda = \sum_{\mu\vdash k} M_{\lambda\mu}(?,m)\, m_\mu$$

$(M_{\lambda\mu}(?,m))_{\lambda,\mu\vdash k}$ is triangular and $M_{\lambda\lambda}(?,m)>0$

using **Remark** $\Rightarrow$ transition matrix invertible $\Rightarrow$ basis

Ex 1 $(n=k=2)$: $(1^2),(2)\vdash 2$, $m_{(1^2)}(X_2)=x_1x_2$

$m_{(2)}(X_2)=x_1^2+x_2^2$

$-\ p_{(1^2)}(X_2)=(x_1+x_2)^2$

$= x_1^2+x_2^2+2x_1x_2 = 2m_{(1^2)}(X_2)+m_{(2)}(X_2)$

$-\ p_{(2)}(X_2)=x_1^2+x_2^2=m_{(2)}(X_2) \rightsquigarrow$

m\p	(1^2)	(2)
(1^2)	2	0
(2)	1	1

Generating Functions

Sunday, January 30, 2022 11:18 AM

Def $e_k(X_n) = \sum \prod$ k distinct $x_i, i \in [n]$

$h_k(X_n) = \sum \prod$ k repeatable $x_i, i \in [n]$

Lemma 5: (a) $E(t) = \sum_{k=0}^{\infty} e_k t^k = \prod_{j=1}^{\infty}(1+x_j t)$

(b) $H(t) = \sum_{k=0}^{\infty} h_k t^k = \prod_{j=1}^{\infty} \frac{1}{1-x_j t}$

Pf: (a) Recall g.f. for partitions w/ distinct parts was

$$\prod_{j=1}^{\infty}(1+x^j) = \sum_{k=0}^{\infty} \left| k = \sum_{\text{all } i \text{ distinct}} i \right| x^k$$

$$\Downarrow$$

$$\prod_{j=1}^{\infty}(1+x_j) = \sum_{k=0}^{\infty} \prod_{\text{all } i \text{ distinct}}^{k} x_i$$

Problem; b4, x was a dummy variable that kept track how large each partition was

Sol Replace $x_j \rightsquigarrow x_j t$, now t keeps track of deg

(b) Similar story with

$$\prod_{j=1}^{\infty} \frac{1}{1-x^j} = \sum_{k=0}^{\infty} \left| k = \sum_{i \text{ can repeat}} i \right| x^k$$

$$\Downarrow$$

$$\prod_{j=1}^{\infty} \frac{1}{1-x_j t} = \sum_{k=0}^{\infty} h_k t^n$$

Prop 6: (a) $e_k(\underbrace{1,\dots,1}_{n \text{ times}}) = \binom{n}{k}$

(b) $e_k(1,2,\dots,n) = \left[{n+1 \atop n+1-k} \right]$ (Stirling of 1st kind)

(c) $e_k(1,q,\dots,q^{n-1}) = q^{\binom{k}{2}} \binom{n}{k}_q$

Prop 7: (a) $h_k(\underbrace{1,\dots,1}_{n\text{ times}}) = \binom{n+k-1}{k}$

(b) $h_k(1,2,\dots,n) = \left\{ {n+k \atop n} \right\}$ (Stirling of 2nd kind)

(c) $h_k(1,q,\dots,q^{n-1}) = \binom{n+k-1}{k}_q$

Identities

Sunday, January 30, 2022 11:18 AM

Remark: $\left[{n \atop k}\right] = \# \pi \in S_n$ s.t. π has exactly k cycles

order matters $\longleftarrow \cdots\cdots \longrightarrow$ distinct

$\underbrace{(ab\;)\cdots(\;)}_{k \text{ cycles}}$

$\left\{{n \atop k}\right\} = \#$ of ways to partition a set of n labeled objects into k non-empty unlabeled subsets

order doesn't matter $\longleftarrow \cdots\cdots \longrightarrow$ repeatable

$\boxed{ab}\;\cdots$

Cor 8: (a) $\displaystyle\sum_{k=1}^{n} \left[{n \atop j}\right] t^j = \prod_{j=0}^{n-1} (t+j)$ $\quad t(t+1)\cdots(t+n-1)$

(b) $\displaystyle\sum_{j=0}^{\infty} \left\{{n+j \atop n}\right\} t^j = \prod_{j=1}^{n} \frac{1}{1-jt}$

Pf: (b) Set $x_j = \begin{cases} j & 1 \le j \le n \\ 0 & j > n \end{cases}$ in Lemma 5 (b). Then use Prop 7 (b), similarly w/ (a)

Cor 9: (q-binomial theorems)

(a) $\displaystyle (1+t)(1+qt)\cdots(1+q^{n-1}t) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q t^k$

(b) $\displaystyle \frac{1}{(1-t)(1-qt)\cdots(1-q^{n-1}t)} = \sum_{j=0}^{\infty} \binom{n+j-1}{j}_q t^k$

Pf (a) Set $x_j = \begin{cases} q^{j-1} & 1 \le j \le n \\ 0 & j > n \end{cases}$ in Lemma 5 (a) and use Prop 6 (c)

Recall $E(-t)H(t) = 1 \Rightarrow$

$$\sum_{j=0}^{n} (-1)^j e_j h_{n-j} = 0 \quad \forall n \ge 1 \;(*)$$

Cor 10: (a) $\displaystyle\sum_{j=0}^{m} (-1)^j \binom{n}{j} \binom{m+n-j-1}{m-j} = 0$

(b) $\displaystyle\sum_{j=0}^{m} (-1)^j \left[{n+1 \atop n+1-j}\right] \left\{{n+m-j \atop n}\right\} = 0$

Schur Polynomials

Sunday, January 30, 2022 11:18 AM

Def Let λ be a partition. A semistandard (Young) tableau of shape λ is a filling of the Young diagram for λ s.t. repeatable

- rows weakly increase (left to right)
- columns strictly increase ← distinct
  - English (top to bottom)
  - French (bottom to top)

Let $SST(\lambda,[n])$ = s.s. tableau of shape λ w/ entries from $[n]$

Def Let λ be a partition. The schur polynomial $s_\lambda(X_n)$ is defined as

"combinatorial def"

$$s_\lambda(X_n) = \sum_{\vec{T}\in SST(\lambda,[n])} \vec{X}^{\vec{T}}$$

$\vec{X}^{\vec{T}} = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$

α_i = # of times i appears in T

Ex 2: $\lambda = (1,1,1) = 1^3,\ n=3 \rightsquigarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix}$ only filling

$\rightsquigarrow s_{(1,1,1)}(X_3) = x_1x_2x_3 = e_3(X_3)$

Exercise Compute

(i) $s_{(3)}(X_3)$ (ii) $s_{(2,1)}(X_3)$

Ans: (i) $\begin{smallmatrix}1&1&1\end{smallmatrix}$, $\begin{smallmatrix}1&1&2\end{smallmatrix}$, $\begin{smallmatrix}1&1&3\end{smallmatrix}$, $\begin{smallmatrix}1&2&2\end{smallmatrix}$, $\begin{smallmatrix}1&2&3\end{smallmatrix}$, $\begin{smallmatrix}1&3&3\end{smallmatrix}$, $\begin{smallmatrix}2&2&2\end{smallmatrix}$, $\begin{smallmatrix}2&2&3\end{smallmatrix}$, $\begin{smallmatrix}2&3&3\end{smallmatrix}$, $\begin{smallmatrix}3&3&3\end{smallmatrix}$

$$s_{(3)}(X_3) = x_1^3+x_2^3+x_3^3+x_1^2(x_2+x_3) + x_2^2(x_1+x_3)+x_3^2(x_1+x_2)+x_1x_2x_3$$

$$= h_3(X_3)$$

(ii) $\begin{smallmatrix}1&1\\2\end{smallmatrix}$ $\begin{smallmatrix}1&1\\3\end{smallmatrix}$ $\begin{smallmatrix}1&2\\2\end{smallmatrix}$ $\begin{smallmatrix}1&2\\3\end{smallmatrix}$ $\begin{smallmatrix}1&3\\2\end{smallmatrix}$ $\begin{smallmatrix}1&3\\3\end{smallmatrix}$ $\begin{smallmatrix}2&2\\3\end{smallmatrix}$ $\begin{smallmatrix}2&3\\3\end{smallmatrix}$

$+2x_1x_2x_3$

$$s_{(2,1)}(X_3) = x_1^2(x_2+x_3)+x_2^2(x_1+x_3)+x_3^2(x_1+x_2)$$

- Notice $p_3(X_3) = s_{(3)}(X_3) - s_{(2,1)}(X_3) + s_{(1,1,1)}(X_3)$

Schur Polynomial 2

Sunday, January 30, 2022 11:18 AM

In general, $P_n = \sum_{j=0}^{n-1} (-1)^j s_{n-j,1^j}$

<u>Def</u> $a_\lambda(X_n) = \det \begin{vmatrix} x_1^{\lambda_1} & x_2^{\lambda_1} & & x_n^{\lambda_1} \\ x_1^{\lambda_2} & x_2^{\lambda_2} & \cdots & \vdots \\ \vdots & \vdots & & \\ x_1^{\lambda_n} & x_2^{\lambda_n} & & x_n^{\lambda_n} \end{vmatrix}$

$\ell(\lambda) \le n$

<u>Thm 11</u>: Let $\ell(\lambda) \le n$. Then

$$s_\lambda(X_n) = \frac{a_{\lambda+\delta_n}(X_n)}{a_{\delta_n}(X_n)}, \quad \delta_n = (n-1, n-2, \dots, 1)$$

"alg def"

<u>Remark</u>: $a_{\delta_n}(X_n) = \prod_{1 \le i < j \le n} (x_i - x_j)$ "Vandermonde determinant"

<u>Ex 3</u>: $\lambda = \begin{array}{|c|}\hline \, \\ \hline \, \\ \hline \, \\ \hline\end{array}$, $n = 3$, $\Rightarrow \lambda + \delta_3 = (3, 2, 1)$

$$a_{\lambda+\delta_3}(X_3) = \begin{vmatrix} x_1^3 & x_2^3 & x_3^3 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \end{vmatrix} =$$

$$= x_1^3(x_2^2x_3 - x_2x_3^2) - x_1^2(x_2^3x_3 - x_2x_3^3) + x_1(x_2^3x_3^2 - x_2^2x_3^3)$$

$$= x_1x_2x_3(x_1^2x_2 - x_1^2x_3) - x_1x_2x_3(x_2^2x_1 - x_1x_3^2) + x_1x_2x_3(x_2^2x_3 - x_2x_3^2)$$

$$= x_1x_2x_3\big(x_1^2(x_2 - x_3) - x_1\underbrace{(x_2^2 - x_3^2)}_{=(x_2+x_3)(x_2-x_3)} + x_2x_3(x_2 - x_3)\big)$$

$$= x_1x_2x_3\big((x_1^2 - x_1x_2 - x_1x_3 + x_2x_3)(x_2 - x_3)\big)$$

$$= x_1x_2x_3\big(\underbrace{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}_{\text{"}a_{\delta_3}(X_3)\text{"}}\big)$$

$$\Rightarrow \frac{a_{\lambda+\delta_3}(X_3)}{a_{\delta_3}(X_3)} = s_{\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline \, \\ \hline\end{array}}(X_3) = e_3(X_3)$$

Combinatorics of Coefficients

Sunday, January 30, 2022 11:18 AM

Lemma 12:

$$h_\lambda(X_n) = \sum_{\vec{T} \in \mathrm{WeakRow}(\lambda,[n])} \vec{x}^{\vec{T}}$$

Prop 13: Let $\lambda, \mu \vdash n$

$$M_{\lambda\mu}(h,m) = \#\left\{ n\times n \text{ matrices } \begin{array}{cc} \begin{pmatrix} a_{11} & & \\ a_{21} & \ddots & \\ & & a_{nn} \end{pmatrix} & \begin{matrix} \mu_1 \\ \vdots \\ \mu_n \end{matrix} \\ \lambda_1 \cdots \lambda_n & \end{array} \;\middle|\; \begin{array}{l} \bullet \text{ Sum of column } i \text{ is } j \\ \bullet \text{ sum of row } j \text{ is } \mu_j \\ \bullet\ a_{ij} \in \mathbb{Z}^{\geq 0} \end{array} \right\}$$

$= S_{\lambda\mu}$

Pf: By def

$$h_\lambda(X_n) = \sum_{\mu \vdash n} M_{\lambda\mu}(h,m)\, m_\mu(X_n)$$

- By "c=d" remark, $M_{\lambda\mu}(h,m)$ is the coefficient of the monomial $\vec{x}^{\vec{\mu}} = x_1^{\mu_1} \cdots x_n^{\mu_n}$
- Lemma 12 $\Rightarrow M_{\lambda\mu}(h,m) = |\mathrm{WeakRow}_\mu(\lambda,[n])|$

writes $h_\lambda(X_n)$ as a sum of monomials

i appears μ_i times

$$\mathrm{WeakRow}_\mu(\lambda,[n]) \rightleftarrows S_{\lambda\mu}$$

$$\Phi : \vec{T} \longmapsto (a_{ij} = \#\text{ of } i\text{'s in row } j)$$

Ex 4:

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 3 & 3 & \\ \cline{1-2} \end{array} \longmapsto \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- $\sum_{j=1}^{n} a_{ij} = \#$ of i's in all rows $= \mu_i$
- $\sum_{i=1}^{n} a_{ij} =$ number of boxes in row j $= \lambda_j$

$\Rightarrow \Phi(\vec{T}) \in S_{\lambda\mu}$. Φ is invertible b/c weakly increasing makes filling unique.

Ex 5:

$$\begin{array}{cc} \begin{pmatrix} 2 & 1 & \\ 1 & 2 & \cdots \\ 0 & 0 & \\ 0 & 0 & \\ 0 & 0 & \\ 0 & 0 & \end{pmatrix} & \begin{matrix} \mu_1 \\ \mu_2 \\ \\ \\ \\ \\ \end{matrix} \\ \begin{matrix} 3 & 3 & \cdots \\ \lambda_1 & \lambda_2 & \end{matrix} & \end{array}$$

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 1 & 2 & 2 \\ \hline \end{array}$$

W.I.:
- 2 1's
- 1 2's

Combinatorics 2

Sunday, January 30, 2022 11:18 AM

<u>Cor 14:</u> $M_{\lambda\mu}(n,m) = M_{\mu\lambda}(n,m)$

<u>Pf:</u> $|\mathbb{S}_{\lambda\mu}| = |\mathbb{S}_{\mu\lambda}|$ f Prop 13

$A \longmapsto A^T$ is a bijection

<u>Upshot:</u> We just used Combinatorics to prove a nontrivial alg fact.

Algebra	Combinatorics
• easier to prove things	• easier to compute things
(computations in algebra are abstract)	(proofs in combinatorics are ad-hoc/on the spot)

Sym funct $\subseteq$ Algebraic Combinatorics (best of both worlds)

Sunday, January 30, 2022 11:18 AM