Monomial and Elementary Symmetric Functions

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1 The Monomial Symmetric Polynomials

Construction of a monomial symmetric polynomial:

- Pick a set of variables $x_1, ..., x_n$
- Pick a monomial in these variables (e.g. $x_\alpha^3 x_\beta$)
- Construct a monomial symmetric polynomial by adding all of the distinct combinations of vaiables of the monomial we chose.

Example 1.1. Find $h \in \Lambda(X_2)$ that includes $x_1^4 x_2$

This polynomial must be invariant under:

$$(12) \longrightarrow x_2^4 x_1, \quad (1)(2) \longrightarrow x_1^4 x_2$$

Hence, we get the following polynomial:

$$h(x_1, x_2) = x_1^4 x_2 + x_2^4 x_1$$

Example 1.2. Find $f \in \Lambda(X_3)$ that includes $x_1^3 x_2$

This polynomial must be invariant under:

$$(12) \longrightarrow x_2^3 x_1, \quad (23) \longrightarrow x_1^3 x_3, \quad (13) \longrightarrow x_3^3 x_2, \quad (123) \longrightarrow x_2^3 x_3, \quad (132) \longrightarrow x_3^3 x_1, \quad (1)(2)(3) \longrightarrow x_1^3 x_2$$

Hence, we get the following polynomial:

$$f(x_1, x_2, x_3) = x_1^3 x_2 + x_1^3 x_3 + x_2^3 x_1 + x_2^3 x_3 + x_3^3 x_1 + x_3^3 x_2$$

Example 1.3. Find $g \in \Lambda(X_3)$ that includes $3x_1^2 x_2 x_3^2$

This polynomial must be invariant under:

$$(12) \longrightarrow 3x_2^2 x_1 x_3^2, \quad (13) \longrightarrow 3x_3^2 x_2 x_1^2, \quad (23) \longrightarrow 3x_1^2 x_3 x_2^2, \quad (123) \longrightarrow 3x_2^2 x_3 x_1^2, \quad (132) \longrightarrow 3x_3^2 x_1 x_2^2,$$
$$(1)(2)(3) \longrightarrow 3x_1^2 x_2 x_3^2$$

We see that there are some terms that simply duplicates, which we do not need. Hence, we get the following polynomial:

$$g(x_1, x_2, x_3) = 3x_1^2 x_2 x_3^2 + 3x_1^2 x_3 x_2^2 + 3x_2^2 x_1 x_3^2$$

Definition 1.4. **(Partition)** $\forall n \in \mathbb{Z}_{\geq 0}$, a partition of n is a weakly decreasing (i.e. $\lambda_n \geq \lambda_{n+1}$) sequence $\{\lambda_j\}_{j=1}^\infty$ of nonnegative integers such that $\sum_{j=1}^\infty \lambda_j = n$. If λ is a partition of n, we write "$\lambda \vdash n$" and $|\lambda| = n$.

Now, we make an important observation: $x_1^3 x_2$ and $x_3^3 x_1$ give the same polynomial

$$x_1^3x_2 + x_1^3x_3 + x_2^3x_1 + x_2^3x_3 + x_3^3x_1 + x_3^3x_2$$

From this observation, we can infer that we can rearrange the factors in our monomial to ensure that when we list the variables in the order $x_1, ..., x_n$ that exponents form a partition. This provides the motivation for the following definition.

Definition. 1.5. Suppose $n \geq 1$ and λ is a partition of n. Then the monomial symmetric polynomial $m_\lambda(X_n)$ indexed by λ is the sum of the monomial $\prod_{j=1}^{l(\lambda)} x_j^{\lambda_j}$ and all of its distinct images under the elements of S_n. Note that $l(\lambda)$ is the length of the partition. Here, we take $x_j = 0$ for all $j > n$, so if $l(\lambda) > n$, then $m_\lambda(X_n) = 0$. This is a fancy way of saying "if there are more exponents in a monomial than the number of variables" then polynomial has to be zero. We also denote $m_{4,4,3,1}(X_n) \longrightarrow$ "$m_{4431}(X_n)$"

Examples 1.6.

- $m_{21}(X_2) = x_1^2x_2 + x_2^2x_1$
- $m_{21}(X_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2$
- $m_{3311}(X_3) = 0$ since $l(\lambda) = 4 > 3 = n$
- $m_{3311}(X_4) = x_1^3x_2^3x_3x_4 + x_1^3x_3^3x_2x_4 + x_1^3x_4^3x_2x_3 + x_2^3x_3^3x_1x_4 + x_2^3x_4^3x_1x_3 + x_3^3x_4^3x_1x_2$

Proposition 1.7. $\forall n \in \mathbb{Z}_{>0}$, $\forall \lambda$ the polynomial $m_\lambda(X_n)$ is a symmetric polynomial in $x_1, ..., x_n$

Proof of proposition. There are two cases:
Case 1. $n < l(\lambda)$, then by definition $m_\lambda(X_n) = 0$ which is obviously a symmetric polynomial.
Case 2. $n \geq l(\lambda)$.

Lemma 1.8. Every permutation is a product of adjacent transpositions.
Proof of lemma. Since every permutation is a product of disjoint cycles, it is sufficient to show that every transposition is a product of adjacent transpositions. If $a_1 < a_2$, then

$$(a_1\ a_2) = \alpha\beta$$
$$\alpha = (a_1, a_1 + 1)(a_1 + 1, a_1 + 2)...(a_2 - 1, a_2)$$
$$\beta = (a_2 - 1, a_2 - 2)(a_2 - 2, a_2 - 3)...(a_1 + 1, a + 1)$$

This proves the lemma. $\square$

With this lemma it suffices to show that $\sigma_j(m_\lambda(X_n)) = m_\lambda(X_n)$, $1 \leq j \leq n-1$, where σ_j is $(j, j+1)$. We want to show that every term $x_1^{\mu_1}...x_n^{\mu_n}$ has the same coefficient in $\sigma_j(m_\lambda(X_n))$ as in $m_\lambda(X_n)$. Now, we observe that the coefficient of $x_1^{\mu_1}...x_n^{\mu_n}$ in $m_\lambda(X_n)$ is 1 if $\mu_1, ..., \mu_n$ is a reordering of λ and 0 otherwise. Hence, we see that if the coefficient of $x_1^{\mu_1}...x_j^{\mu_j}x_{j+1}^{\mu_{j+1}}...x_n^{\mu_n}$ is 1, then the coefficient of $x_1^{\mu_1}...x_{j+1}^{\mu_j}x_j^{\mu_{j+1}}...x_n^{\mu_n}$ is 1 and vice versa. This proves proposition 1.7. $\square$

Remark 1.9. If $\lambda \vdash k$, then the monomial symmetric polynomial $m_\lambda(X_n)$ is homogeneous of degree k, so $m_\lambda(X_n) \in \Lambda_k(X_n)$.

Remark 1.10. Homogeneous: $x^5 + 2x^3y^2 + 9xy^4$, Non-homogeneous: $x^3 + 3x^2y^4 + z^7$

Proposition 1.11. If $n \geq 1$, $k \geq 0$, and $n \geq k$, then the set

$$\{m_\lambda(X_n) : \lambda \vdash k\}$$

of monomial symmetric polynomials is a basis for $\Lambda_k(X_n)$. In particular, dim $\Lambda_k(X_n) = p(k)$, the number of partitions of k.

Proof of proposition. To prove that the given set is a basis, it suffices to check two things: linear independence, and span.

Linear Independence: First, observe that if λ and μ are partitions with $\lambda \neq \mu$, then $m_\lambda(X_n)$ and $m_\mu(X_n)$ have no terms in common. Therefore, since each $m_\lambda(X_n)$ is nonzero, $\sum_{\lambda \vdash k} a_\lambda m_\lambda(X_n) = 0$ can only occur if $a_\lambda = 0$ for all $\lambda \vdash k$. Hence, $\{m_\lambda(X_n) : \lambda \vdash k\}$ is linearly independent.

Span: Suppose $f \in \Lambda_k(X_n)$. If $f = 0$, then $f = \sum_{\lambda \vdash k} 0 \cdot m_\lambda(X_n)$, so suppose $f \neq 0$. Now, argue by induction on the number of terms of f. f has a term of the form $\alpha \prod_{j=1}^{n} x_j^{\mu_j}$ for some $\mu \vdash k$ and some constant α. Then $f - \alpha m_\mu(X_n) \in \Lambda_k(X_n)$ and $f - \alpha m_\mu(X_n)$ has fewer terms than f. By induction, $f - \alpha m_\mu(X_n)$ is a linear combination of the elements of $\{m_\lambda(X_n) : \lambda \vdash k\}$, and thus f is as well.

There is exactly one monomial symmetric polynomial in $\Lambda_k(X_n)$ for each partition of k, which yields $\dim \Lambda_k(X_n) = p(k)$. □

2 The Elementary Symmetric Functions

Motivation: Suppose we have a polynomial $f(z)$ with 1 as the leading coefficient (monic) and roots $x_1, ..., x_n$. Then we can express f as

$$f(z) = (z - x_1)(z - x_2)...(z - x_n)$$

Notice that permuting x_i does not change f and if we expand f we get that the coefficient of z^k is a symmetric polynomial in $x_1, ..., x_n$. More specifically, for any k with $0 \le k \le n$, the coefficient of z^{n-k} is the sum of all products of exactly k distinct x_j's up to sign. This is the elementary symmetric polynomial of degree k in $x_1, ..., x_n$.

Definition 2.2. $\forall n, k \in \mathbb{Z}_{>0}$, the elementary symmetric polynomial $e_k(X_n)$ is given by

$$e_k(X_n) = \sum_{1 \le j_1 < ... < j_k \le n} \prod_{m=1}^{k} x_{j_m} = \sum_{\substack{J \subseteq [n] \\ |J| = k}} \prod_{j \in J} x_j$$

while the elementary symmetric function e_k is given by

$$e_k = \sum_{\substack{J \subseteq \mathbb{Z}_{>0} \\ |J| = k}} \prod_{j \in J} x_j$$

By convention $e_0(X_n) = 1$ and $e_0 = 1$, and if $n < k$, then $e_k(X_n) = 0$.

Remark 2.3. $(-1)^k e_k(X_n)$ is the coefficient of z^{n-k} in the polynomial $(z - x_1)...(z - x_n)$.

Definition 2.4. Suppose $n \ge 1$ and λ is a partition. Then the elementary symmetric polynomial indexed by λ, $e_\lambda(X_n)$, is given by

$$e_\lambda(X_n) = \prod_{j=1}^{l(\lambda)} e_{\lambda_j}(X_n)$$

Similarly, the elementary symmetric function indexed by λ, e_λ, is given by

$$e_\lambda = \prod_{j=1}^{l(\lambda)} e_{\lambda_j}$$

Also, note that if $n < \lambda_1$, then $e_\lambda(X_n) = 0$.

$\forall n \ge k \ge 1$, we can write $e_k(X_n)$ as a linear combination of monomial symmetric polynomials, and then write e_k as a linear combination of monomial symmetric functions.

Example 2.5. Write $e_{21}(X_3)$, $e_{21}(X_4)$, and $e_{21}(X_5)$ as linear combinations of monomial symmetric polynomials, and then write e_{21} as linear combination of monomial symmetric functions.

We get

- $e_{21}(X_3) = (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2x_3 + x_1x_2^2 + x_1x_2x_3 + x_2^2x_3 + x_1x_2x_3 + x_1x_3^2 + x_2x_3^2 = 3m_{111}(X_3) + m_{21}(X_3)$

- Similarly, $e_{21}(X_4) = 3m_{111}(X_4) + m_{21}(X_4)$
- And $e_{21}(X_5) = 3m_{111}(X_5) + m_{21}(X_5)$
- Hence, $e_{21} = 3m_{111} + m_{21}$

Remark 2.6. We will define the following notation: For any partitions λ and μ with $|\lambda| = |\mu|$ and for any $n \geq 1$, let $M_{\lambda,\mu,n}(e,m)$ be the rational number defined by

$$e_\lambda(X_n) = \sum_{\mu \vdash |\lambda|} M_{\lambda,\mu,n}(e,m) m_\mu(X_n)$$

Similarly, $M_{\lambda,\mu}(e,m)$ is the rational number defined by

$$e_\lambda = \sum_{\mu \vdash |\lambda|} M_{\lambda,\mu}(e,m) m_\mu$$

Proposition 2.7. For any partitions λ, μ with $|\lambda| = |\mu|$, if $n \geq |\lambda|$, then $M_{\lambda,\mu,n}(e,m) = M_{\lambda,\mu}(e,m)$. In particular, if $n \geq |\lambda|$, then $M_{\lambda,\mu,n}(e,m)$ is independent of n.

Proof of proposition. Set $k = l(\mu)$ and $l = l(\lambda)$. The coefficient $M_{\lambda,\mu}(e,m)$ is the coefficient of $x_1^{\mu_1} ... x_k^{\mu_k}$ in e_λ. If $n \geq |\lambda|$, then this coefficient is determined by those terms in $e_{\lambda_1}, ..., e_{\lambda_l}$, involving only $x_1, ..., x_k$. Since $k \leq |\lambda| \leq n$, the variables $x_1, ..., x_k \in \{x_1, ..., x_n\}$, and $M_{\lambda,\mu}(e,m)$ is determined by those terms in $e_{\lambda_1}, ..., e_{\lambda_l}$ involving only $x_1, ..., x_n$. This is also how $M_{\lambda,\mu,n}(e,m)$ is determined, so $M_{\lambda,\mu}(e,m) = M_{\lambda,\mu,n}(e,m)$ $\square$

Remark 2.8. To make our work easier, we represent each term of e_k with a filling of a $1 \times k$ tile with distinct positive integers, in increasing order from left to right, corresponding to the subscripts of the factors in that term. For example, the term $x_2x_3x_5x_7$ corresponds to the filling of a 1×4 tile. By stacking and left-justifying these fillings, we can represent each term as a filling of the Ferrers diagram, in which the entries in each row are strictly increasing from left to right. e.g.

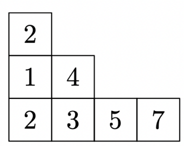

Figure 2.9. The filling corresponding to the product of $x_2x_3x_5x_7$, x_1x_4, and x_2

Similarly, for $\mu = (2^2, 1^3)$, the Ferrers diagram for $(4,2,1)$ becomes

"Do the example - write Ferrers diagram - Put 1s (2) and 2s (2)" - put 3,4,5 - count number of possibilities

This gives the coefficient of m_{22111} in e_{421}, which is 11.

Definition 2.10. Suppose λ and μ are partitions. We say λ is greater than μ in lexicographic order, and we write $\lambda >_{\text{lex}} \mu$, whenever there is a positive integer m such that $\lambda_j = \mu_j$ for $j < m$ and $\lambda_m > \mu_m$. Here we take $\lambda_j = 0$ if $j > l(\lambda)$ and we take $\mu_j = 0$ if $j > l(\mu)$.

Remark 2.11. This is very similar to the order we use for the alphabet.

Example 2.12. Write the partitions of 6 in lexicographic order, from largest to smallest.

$\Rightarrow (6) >_{\text{lex}} (5,1) >_{\text{lex}} (4,2) >_{\text{lex}} (4,1^2) >_{\text{lex}} (3^2) >_{\text{lex}} (3,2,1) >_{\text{lex}} (3,1^3) >_{\text{lex}} (2^3) >_{\text{lex}} (2^2,1^2) >_{\text{lex}} (2,1^4) >_{\text{lex}} (1^6)$

Proposition 2.13. Suppose λ, μ are partitions with $|\lambda| = |\mu|$. Then

(i) if $\mu >_{\text{lex}} \lambda'$ then $M_{\lambda,\mu}(e,m) = 0$;

(ii) $M_{\lambda,\lambda'}(e,m) = 1$.

Proof of proposition. (i) We have

$$e_\lambda = e_{\lambda_1} ... e_{\lambda_{l(\lambda)}}$$

and we note that $M_{\lambda,\mu}(e,m)$ is the coefficient of $x_1^{\mu_1} ... x_{l(\mu)}^{\mu_{l(\mu)}}$ in this product. If $\mu >_{\text{lex}} \lambda'$, then by definition there exists $m \geq 1$ such that $\mu_m > \lambda'_m$ and $\mu_j = \lambda'_j$ for $1 \leq j < m$. Each factor e_{λ_j} can contribute at most one factor x_1 to our term, so $\mu_1 = \lambda'_1$ implies each factor e_{λ_j} contributes exactly one factor x_1. (This corresponds to filling the first column of the Ferrers diagram of λ with 1's.) Similarly, only those e_{λ_j} with $\lambda_j \geq 2$ can contribute a factor x_2, so each such e_{λ_j} must contribute exactly one factor x_2. (This corresponds to filling the second column of the Ferrers diagram of λ with 2's.) Proceeding in this way, we see that only those e_{λ_j} with $\lambda_j \geq m$ can contribute a factor x_m to our term, so the exponent on x_m is at most λ'_m. Since $\mu_m > \lambda'_m$, the term $x_1^{\mu_1} ... x_{l(\mu)}^{\mu_{l(\mu)}}$ does not appear in our product, and 1 the result follows.

(ii) Arguing as in the proof of (i), we see the only way to produce the term $x_1^{\lambda'_1} ... x_{l(\lambda')}^{\lambda'_{l(\lambda')}}$ is to choose the term $x_1 ... x_{l(\lambda')}$ e_{λ_j} for all j. Now the result follows. □

Remark 2.14. The converse is false.

Example 2.15. Write e_{31} as a linear combination of monomial symmetric functions.

- Assume $x_1, ..., x_4$
- Normally, $\binom{4}{3}\binom{4}{1} = 16$ terms
- Use Ferrers, diagram (3,1) and plug in m_{1^4}, $m_{2,1^2}$, m_{2^2}, m_{31} to find the coefficients

Corollary 2.16. The set $\{e_\lambda | \lambda \vdash k\}$ of elementary symmetric functions is a basis for Λ_k.

Proof of corollary. Let A be the $p(k) \times p(k)$ matrix whose rows and columns are indexed by the partitions of k, in lexicographic order from smallest to largest, and whose entries are given by $A_{\lambda\mu} = M_{\lambda',\mu}(e,m)$. By the previous proposition, A is a lower triangular matrix whose diagonal entries are all equal to 1, so det $A = 1$ and A is invertible. Since $e_{\lambda'} = \sum_{\mu \vdash k} A_{\lambda\mu} m_\mu$, each monomial symmetric function m_μ is a linear combination of elementary symmetric functions, and $\{e_\lambda | \lambda \vdash k\}$ spans Λ_k by a previous proposition. But dim $\Lambda_k = p(k) = |\{e_\lambda | \lambda \vdash k\}|$, so $\{e_\lambda | \lambda \vdash k\}$ must also be linearly independent. Therefore $\{e_\lambda | \lambda \vdash k\}$ is a basis, which is what we wanted to prove. □

Fact 2.17. The following hold for all partitions $\lambda, \mu \vdash k$.
$M_{\lambda,\mu}(e,m)$ is the number of $k \times k$ matrices in which every entry is 0 or 1,the sum of the entries in row m is μ_m for all m, and the sum of the entries in column j is λ_j for all j.

Corollary 2.18. For all partitions $\lambda, \mu \vdash k$, we have

$$M_{\lambda,\mu}(e,m) = M_{\mu,\lambda}(e,m)$$

Proof of corollary. For any partitions $\lambda, \mu \vdash k$, let $B_{\lambda,\mu}$ be the set of $k \times k$ matrices in which every entry is 0 or 1, the sum of the entries in row m is μ_m for all m, and the sum of the entries in column j is λ_j for all j. By the previous fact we have $|B_{\lambda,\mu}| = M_{\lambda,\mu}(e,m)$. The result follows from the fact that the transpose map is a bijection between $B_{\lambda,\mu}$ and $B_{\mu,\lambda}$. □

Proposition 2.19. The ordinary generating function for the sequence $\{e_n\}_{n=0}^\infty$ of elementary symmetric functions is

$$\sum_{n=0}^{\infty} e_n t^n = \prod_{j=1}^{\infty} (1 + x_j t)$$

We often write $E(t)$ to denote this generating function.

Proof of proposition. We can build each elementary symmetric function en uniquely by adding the terms which result from deciding, for each j, whether to include x_j as a factor or not. This matches our computation of the product on the right hand side of the equality: we construct each term by deciding, for each factor $1 + x_j t$, whether to use 1 or $x_j t$ as a factor. $\square$

3 References

References

[1] Eric S. Egge (2019) *An Introduction to Symmetric Functions and Their Combinatorics*, American Mathematical Society