Lie Superalgebras: Fundamentals

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1 Definitions

Definition 1.1. *A super vector space is a $\mathbb{Z}_2-$graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Given $a \in V_i$, let the parity be $|a| = i$, $i \in \mathbb{Z}_2$.*

Given a super vector space V, let Π be the parity reversing functor where $\Pi(V)_{\bar{i}} = V_{\overline{i+1}}$ for $i \in \mathbb{Z}_2$.

Definition 1.2. *A Lie superalgebra is a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a $\mathbb{Z}_2-$graded bilinear operation $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that for all homogeneous elements a, b, c*

(1) Skew-supersymmetry: $[a,b] = -(-1)^{|a|\cdot|b|}[b,a]$

(2) Super Jacobi identity: $[a,[b,c]] = [[a,b],c] + (-1)^{|a|\cdot|b|}[b,[a,c]]$

Remark. If $\mathfrak{g} = \mathfrak{g}_0$ is completely even we recover the definition of a lie algebra.

Example 1. Let A be an associative superalgebra. Then $(A, [-,-]_s)$ is a Lie superalgebra where

$$[a,b]_s = ab - (-1)^{|a||b|}ba$$

Definition 1.3. *A map $f : \mathfrak{g} \to \mathfrak{h}$ between lie superalgebras is a homomorphism if f is even and*

$$f([a,b]) = [f(a), f(b)]$$

Example 2. Let $\mathfrak{g}$ be a lie superalgebra, then $\mathrm{End}(\mathfrak{g})$ is a lie superalgebra by Example 1. The adjoint representation of $\mathfrak{g}$ is the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$

$$\mathrm{ad}_a(b) := [a,b]$$

which is a homomorphism by the super jacobi identity.

Remark. Since $[-,-]$ is $\mathbb{Z}_2-$graded we see that $\mathrm{ad}|_{\mathfrak{g}_0} : \mathfrak{g}_0 \to \mathrm{End}(\mathfrak{g}_1)$, aka $\mathfrak{g}_1$ is a $\mathfrak{g}_0$ module.

Example 3 (general linear lie superalgebra)**.** Let $V = V_0 \oplus V_1 \cong \mathbb{C}^{m|n}$ (where $m = \dim V_0, n = \dim V_1$) be a super vector space. Then $\mathfrak{gl}(m|n) := (\mathrm{End}(\mathbb{C}^{m|n}), [-,-])$ from Example 1. Fixing a basis, we see that $\mathfrak{gl}(m|n)$ consists of block matrices of the form

$$\begin{array}{r} \\ m\{ \\ n\{ \end{array}\!\!\begin{array}{c} \overbrace{\quad}^{m}\overbrace{\quad}^{n} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array} \tag{1}$$

Explicitly,

$$\mathfrak{gl}(m|n)_0 = \begin{array}{r} \\ m\{ \\ n\{ \end{array}\!\!\begin{array}{c} \overbrace{\quad}^{m}\overbrace{\quad}^{n} \\ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \end{array}, \qquad \mathfrak{gl}(m|n)_1 = \begin{array}{r} \\ m\{ \\ n\{ \end{array}\!\!\begin{array}{c} \overbrace{\quad}^{m}\overbrace{\quad}^{n} \\ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \end{array}$$

We then have that $\mathfrak{gl}(m|n)_0 \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ while $\mathfrak{gl}(m|n)_1 \cong \left(\mathbb{C}^m \otimes \mathbb{C}^{n^*}\right) \oplus \left(\mathbb{C}^{m^*} \otimes \mathbb{C}^n\right)$ as a $\mathfrak{gl}(m|n)_0$ module.

Example 4 (special linear lie superalgebra)**.** Given an element $g \in \mathfrak{gl}(m|n)$ in the form Eq. (1), define the supertrace as

$$\operatorname{str}(g) = \operatorname{tr}(A) - \operatorname{tr}(D)$$

Facts:

(1) $\operatorname{str}([g,h]_s) = 0\ \forall g,h \in \mathfrak{gl}(m|n)$.

(2) The subspace

$$\mathfrak{sl}(m|n) := \{g \in \mathfrak{gl}(m|n) |\operatorname{str}(g) = 0\}$$

is a lie subsuperalgebra of $\mathfrak{gl}(m|n)$.

(3) $[\mathfrak{gl}(m|n), \mathfrak{gl}(m|n)] = \mathfrak{sl}(m|n)$.

Definition 1.4. *A bilinear form $\langle -,- \rangle$ on a super vector space $V = V_0 \oplus V_1$ is supersymmetric if*

$$\langle v, w\rangle = (-1)^{|v||w|} \langle w, v\rangle$$

It is said to be even if $\langle V_0, V_1\rangle = 0$.

Lemma 1.5. *$\mathfrak{gl}(m|n)$ and $\mathfrak{sl}(m|n)$ (Except $(m,n) = (1,1),(2,1)$) are basic lie superalgebras meaning that they admit non-degenerate even supersymmetric bilinear forms.*

Proof. $\langle a,b\rangle = \operatorname{str}(ab)$ does the trick. Call this the supertrace form. ■

Definition 1.6. *Given a <u>basic</u> lie superalgebra $\mathfrak{g}$, a cartan subalgebra $\mathfrak{h}$ is defined to be a Cartan subalgebra of the even subalgebra $\mathfrak{g}_0$ and the Weyl group of $\mathfrak{g}$ is defined to be the Weyl group of $\mathfrak{g}_0$.*

Example 5. The Cartan subalgebra for $\mathfrak{gl}(m|n)$ will be the Cartan subalgebra for $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ aka diagonal matrices in $\mathfrak{gl}(m+n)$. Namely let $I(m|n) = \left\{\overline{1}, \ldots, \overline{m}, 1, \ldots, n\right\}$ with total order

$$\overline{1} < \ldots < \overline{m} < 0 < 1 < \ldots < n$$

Then $\mathfrak{h} = \bigoplus_{i \in I(m|n)} \mathbb{C}E_{ii}$. Note

$$\langle E_{ii}, E_{jj}\rangle = \begin{cases} 1 & \text{if } \overline{1} \le i = j \le \overline{m} \\ -1 & \text{if } 1 \le i = j \le n \\ 0 & \text{if } i \neq j \end{cases}$$

Definition 1.7. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, which is <u>basic</u>. For $\alpha \in \mathfrak{h}^*$, let*

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [h,g] = \alpha(h)g,\ \forall h \in \mathfrak{h}\}$$

Then the root system for $\mathfrak{g}$ is defined to be

$$\Phi = \{\alpha \in \mathfrak{h}^* | \mathfrak{g}_\alpha \neq 0, \alpha \neq 0\}$$

And define the even and odd roots to be

$$\Phi_0 := \{\alpha \in \Phi | \mathfrak{g}_\alpha \cap \mathfrak{g}_0 \neq 0\} \qquad \Phi_1 := \{\alpha \in \Phi | \mathfrak{g}_\alpha \cap \mathfrak{g}_1 \neq 0\}$$

Theorem 1.8. *Let $\mathfrak{g}$ be a basic lie superalgebra with a Cartan subalgebra $\mathfrak{h}$. Then*

(1) We have a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

(2) $\langle -, - \rangle |_{\mathfrak{h}}$ *is non-degenerate and* $W-$*invariant.*

(3) $\dim \mathfrak{g}_\alpha = 1$ *for* $\alpha \in \Phi$ *(this relies on non-degeneracy)*[1].

(4) Φ, Φ_0, Φ_1 *are each invariant under the action of* W *on* $\mathfrak{h}^*$.

Example 6 (Root system for $\mathfrak{gl}(m|n)/\mathfrak{sl}(m|n)$)**.** Because the cartan for $\mathfrak{gl}(m|n)$ is contained in the even part, the super lie bracket reduces to the usual lie bracket for the action of the cartan on $\mathfrak{gl}(m|n)$. Hence, the roots of $\mathfrak{gl}(m|n)$ are the same as the roots of $\mathfrak{gl}(m+n)$ as a set but we have now partitioned them into even and odd roots. Specifically, let $\{\delta_i, \epsilon_j\}_{i,j} \subset \mathfrak{h}^*$ be the dual basis to $\{E_{\overline{i}\overline{i}}, E_{jj}\}$ under $\langle -, - \rangle$ The root system for $\mathfrak{gl}(m|n)/\mathfrak{sl}(m|n)$ is given by

$$\begin{aligned}
\Phi_0 &= \{\epsilon_i - \epsilon_j \,|i \neq j \in I(m|n), i,j > 0 \text{ or } i,j < 0\} \\
\Phi_1 &= \{\delta_i - \epsilon_j, \epsilon_k - \delta_\ell | i,j \in I(m|n), 1 \le i, \ell \le m, 1 \le j,k \le n\}
\end{aligned}$$

[Draw on block matrices] Because $\mathfrak{h} \cong \mathfrak{h}^*$ under the map $h \mapsto \langle h, - \rangle$ we now have a non-degenerate bilinear form $(-,-)$ on $\mathfrak{h}^*$. Using the results in Example 5 we see that

$$(\delta_i, \delta_j) = \delta_{ij}, \qquad (\epsilon_i, \epsilon_j) = -\delta_{ij}, \qquad (\epsilon_k, \delta_\ell) = 0$$

Definition 1.9. *A root* $\alpha \in \Phi$ *is called isotropic if* $(\alpha, \alpha) = 0$[2]. *Let* $\overline{\Phi_1}$ *denote the set of isotropic odd roots.*

Isotropic roots are necessarily odd, as even roots are roots of $\mathfrak{g}_0$ a regular lie algebra. Assuming $\mathfrak{g}_0 = \mathfrak{g}_{ss} \oplus \mathfrak{a}$ is reductive, where $\mathfrak{g}_{ss}$ is the semisimple part and $\mathfrak{a}$ is the abelian part. But since $\mathfrak{a}$ is abelian it's contained in the 0 root space for the Cartan. Roots are nonzero and thus the roots of a reductive lie algebra coincide with the roots of the semisimple part $\mathfrak{g}_{ss}$ which is a direct sum of simple lie algberas. For simple lie algebras any symmetric invariant form (in particular the supertrace form restricted to $\mathfrak{g}_0$) is a nonzero scalar multiple of the Killing form. But for lie algebras the Killing form is positive definite on the $\mathbb{Q}-$span of Φ and thus $(\alpha, \alpha) > 0$ for even roots.

Example 7. In $\mathfrak{gl}(1|1)$ consider the odd root $\delta_1 - \epsilon_1$. We calculate that

$$(\delta_1 - \epsilon_1, \delta_1 - \epsilon_1) = (\delta_1, \delta_1) + (\epsilon_1, \epsilon_1) = 0$$

Remark. Because $(\alpha, \alpha) = 0$ for some roots, drawing roots for lie superalgebras is slightly dangerous as angles and size no longer tell us any algebraic information. However for $\mathfrak{gl}(m|n)$ we will draw the roots as if they were roots of $\mathfrak{gl}(m+n)$ and indicate which roots are isotropic, etc.

2 Positive Roots

Definition 2.1. *For* $\mathfrak{g}$ *a basic lie algebra define*

$$\Phi^+(H) = \{\alpha \in \Phi \,|\, \langle H, \alpha \rangle_K > 0\}$$

where $\langle -, - \rangle_K$ *is the usual killing form on* $\mathfrak{g}$ *and* H *is a hyperplane not containing any of the roots. Let* $\Delta(H)$ *be the set of simple roots of* $\Phi^+(H)$.

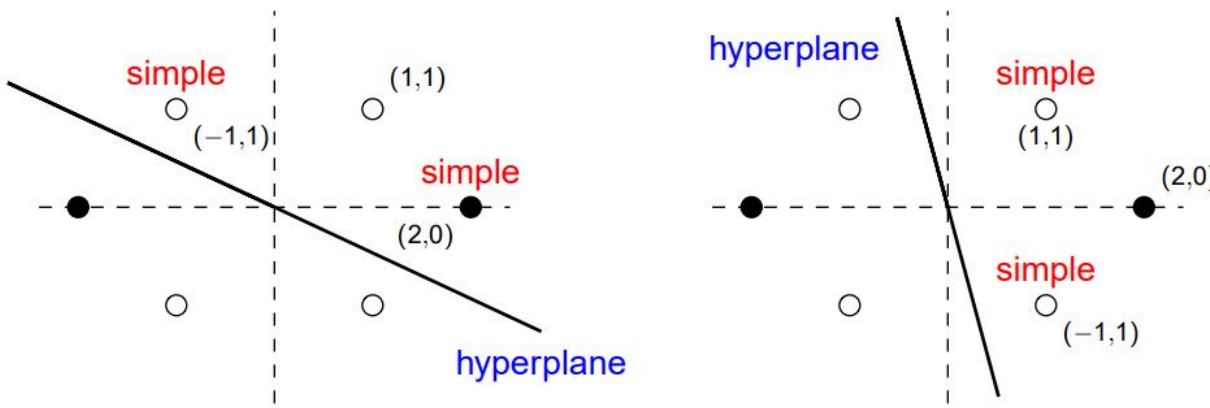
[1] Like affine lie algebras, imaginary root spaces need not be 1-dimensional. Root spaces of $q(n)$ are $(1|1)-$ dimensional for instance.

[2] This also occurs for imaginary roots for affine lie algebras.

Warning. The choice of H matters now as different choices may not be conjugate to each other under the action of the Weyl group. Consider $\mathfrak{gl}(2|1)$.

$$\left(\begin{array}{cc|c} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ \hline 0 & 0 & \epsilon_3 \end{array}\right)$$

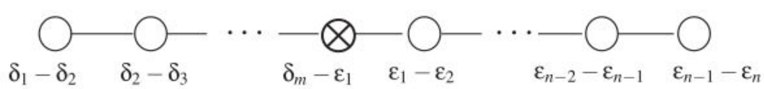
We then see that we have one even root $\epsilon_1 - \epsilon_2$ and two odd isotropic roots $\epsilon_2 - \epsilon_3, \epsilon_1 - \epsilon_3$. See below for two different choices of positive roots.



[The black simple root in left diagram is the even root, draw even roots with ○ draw odd isotropic roots with ⊗, and non-isotropic odd roots with ●]. The corresponding decorated Dynkin diagrams will be

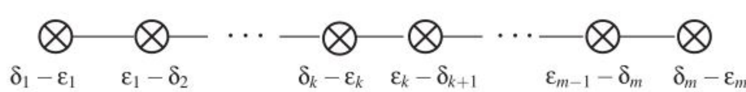
As the Weyl sends even roots to even roots and odd roots to odd roots, the two choices of simple roots above are not conjugate to each other under W.

Example 8 (Standard Simple Roots for $\mathfrak{gl}(m|n)$)**.** Using the notation from Example 6, the standard simple roots(fundamental system) for $\mathfrak{gl}(m|n)$ is given by



We note that the (super)lengths of the roots after the isotropic odd roots are -2.

Example 9 (Nonstandard Simple Roots)**.** If $n = m$, then $\mathfrak{gl}(n|n)$ has the following fundamental system consisting of all isotropic odd roots



Given H, define

$$\mathfrak{n}^+(H) = \bigoplus_{\alpha \in \Phi^+(H)} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^-(H) = \bigoplus_{\alpha \in \Phi^-(H)} \mathfrak{g}_\alpha$$

$\mathfrak{b}(H) = \mathfrak{h} \oplus \mathfrak{n}^+(H)$ is called a Borel subalgebra of $\mathfrak{g}$ corresponding to H. $\mathfrak{b}(H)$ is solvable, however,

Warning. Unlike in the usual case, Borel subalgebras for lie superalgebras need not be MAXIMAL solvable subalgebras. First, we have seen that $\mathfrak{sl}(1|1)$ only has one root, and by Example 7 it is isotropic. In fact the converse is true, aka

Lemma 2.2. *Let α be an isotropic odd root. Let $e_\alpha \in \mathfrak{g}_\alpha$. Then*

$$\mathbb{C}e_\alpha \oplus \mathbb{C}e_{-\alpha} \oplus \mathbb{C}[e_\alpha, e_{-\alpha}] \cong \mathfrak{sl}(1|1)$$

Now, $\mathfrak{sl}(1|1)$ consists of matrices of the form

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix}$$

and this will actually make $\mathfrak{sl}(1|1)$ solvable (a instead of $-a$ in the bottom right corner will make things in the bottom left corner cancel out)! Thus, given a Borel subalgebra $\mathfrak{b}(H)$ and an isotropic odd root α, the subalgebra $\mathfrak{b}(H) \oplus \mathfrak{g}_{-\alpha}$ is solvable[3] so $\mathfrak{b}(H)$ is not maximal solvable.

3 Odd Reflections

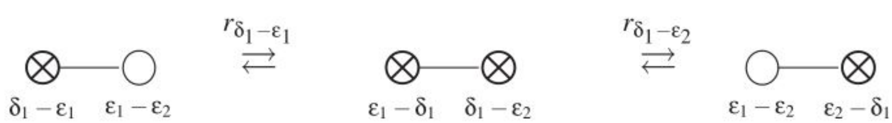
Lemma 3.1 (Serganova)**.** *Let $\mathfrak{g}$ be a basic Lie superalgebra and let Δ be a fundamental system for Φ^+. Let α be an odd isotropic root. Define the odd reflection r_α acting on Δ by*

$$r_\alpha(\beta) = \begin{cases} \beta + \alpha & \text{if } (\beta, \alpha) \neq 0 \\ \beta & \text{if } (\beta, \alpha) = 0, \beta \neq \alpha \\ -\alpha & \text{if } \beta = \alpha \end{cases}$$

Then $\Delta_\alpha := r_\alpha(\Delta)$ is a fundamental system for the set of positive roots $\Phi^+_\alpha := \{-\alpha\} \cup \Phi^+ \setminus \{\alpha\}$.

Remark. Because $(\alpha, \alpha) = 0$, r_α doesn't come from usual formula for reflections about hyperplane perpendicular to α. In fact, r_α need not extend to a linear map $\mathfrak{h}^* \to \mathfrak{h}^*$!

Example 10. In $\mathfrak{gl}(1|2)$ starting with the standard fundamental system, we can apply odd reflections to get the other 2 as seen below



Let us compute how to go from the standard fundamental system to the one in the middle. Let $\alpha = \delta_1 - \epsilon_1$. Then according to the formula for Π_α, we need to compute the bilinear form of α with all roots of Π

- We compute that $(\delta_1 - \epsilon_1, \epsilon_1 - \epsilon_2) = -(\epsilon_1, \epsilon_1) = 1$.
- Thus we end up with the root $\epsilon_1 - \epsilon_2 + \delta_1 - \epsilon_1 = \delta_1 - \epsilon_2$.
- $\alpha \mapsto -\alpha = \epsilon_1 - \delta_1$ is the other simple root in Π_α.

Remark. Note that odd reflections are all "simply laced" reflections. The formula for the odd reflection is the same exactly when the simple root α has at most one line with other roots β in the Dynkin diagram.

Definition 3.2. *Given a Borel subalgebra $\mathfrak{b}$ and an isotropic odd root α, define*

$$\mathfrak{b}^\alpha = \mathfrak{h} \oplus \bigoplus_{\beta \in \Phi^+_\alpha} \mathfrak{g}_\beta$$

[3] The sum of two solvable subalgebras is solvable.

4 Misc

- Levi's theorem isn't true.
- Lie's theorem isn't true.
- Semisimple lie superalgebras are not direct sum of simple lie superalgebras.