§1. Gromov-Witten Theory

§1.1 Moduli Spaces of Stable Maps

X nonsingular projective variety $/\mathbb{C}$

<u>Definition</u> (stable map)

$g, n \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X;\mathbb{Z})$ effective curve class

A genus g, n-pointed, degree β prestable map to X is a morphism $u : (C, x_1, \ldots, x_n) \to X$, where

(1) C is a connected projective curve with at most nodal singularities, $h^1(C, \mathcal{O}_C) := \dim H^1(C, \mathcal{O}_C) = g$

(2) $x_1, \ldots, x_n$ distinct smooth points on C

$(C, x_1, \ldots, x_n)$ is a genus g, n-pointed prestable curve

(3) $u_*[C] = \beta$

isomorphism:

$$\begin{tikzcd}
(C, x_1, \ldots, x_n) \arrow[d, "\phi", "\wr"'] \arrow[dr, "u"] & \\
(C', x_1', \ldots, x_n') \arrow[r, "u'"'] & X
\end{tikzcd}$$

A genus g, n-pointed, deg β prestable map is stable if its automorphism group is finite.

$\overline{\mathcal{M}}_{g,n}(X,\beta)$ moduli of genus g, n-pointed, degree β stable maps to X

<u>Fact</u> $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a <u>proper</u> <u>Deligne-Mumford stack</u>

Compact Hausdorff (usually) singular orbifold

<u>The evaluation maps</u> $i=1,\dots,n$

$$ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow X$$

$$[u:(C,\underset{\substack{\shortparallel \\ (x_1,\dots,x_n)}}{\underline{x}}) \to X] \longmapsto u(x_i)$$

<u>The universal curve and the universal map</u>

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n+1}(X,\beta) & \xrightarrow{ev_{n+1}} & X \\ \downarrow \pi \uparrow \sigma_i & & \\ \overline{\mathcal{M}}_{g,n}(X,\beta) & & \end{array}$$

$\pi \leftarrow$ forget the $(n+1)$-th marked point

σ_i, $i=1,\dots n$

$$\begin{array}{ccc} C & \xrightarrow{u} & X \\ \downarrow & & \\ \xi = [(C,\underline{x}),u] & & \end{array}$$

$\sigma_i(\xi) = x_i$

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{X}} = \overline{\mathcal{M}}_{g,n+1}(X,\beta) & \longrightarrow & \mathcal{C}_{\mathfrak{M}} = \mathfrak{M}^{pre}_{g,n+1} \\ \downarrow \pi_{\mathcal{X}} & \square & \downarrow \pi_{\mathfrak{M}} \\ \mathcal{X} := \overline{\mathcal{M}}_{g,n}(X,\beta) & \xrightarrow{\text{forget}} & \mathfrak{M} = \mathfrak{M}^{pre}_{g,n} \\ [(C,\underline{x}), u] & \longmapsto & [(C,\underline{x})] \end{array}$$

moduli of genus g n-pointed prestable curves

smooth Artin stack of $\dim_{\mathbb{C}}$ $3g-3+n$

$$\mathfrak{M}^{pre}_{g,n} = \bigsqcup_{\Gamma \in G^{pre}_{g,n}} \mathfrak{M}_\Gamma$$ disjoint union of infinitely many strata

U open

$$\overline{\mathcal{M}}_{g,n} = \bigsqcup_{\Gamma \in G_{g,n}} \mathcal{M}_\Gamma$$ disjoint union of finitely many strata

$\mathcal{M}_\Gamma = \left[\left(\prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}\right) / \mathrm{Aut}(\Gamma)\right]$ $\quad G_{g,n} \subset G^{pre}_{g,n}$

partially ordered set

proper smooth DM stack of $\dim_{\mathbb{C}}$ $3g-3+n$
(Compact complex orbifold)

A. Giacchetto's lectures

$\overline{\mathcal{M}}_{g,n}$ is nonempty $\iff$ $2g-2+n>0$

$\mathcal{M}_\Gamma = \left[\left(\prod_{v \in V(\Gamma)} \mathcal{M}_{g(v),n(v)}(X,\beta(v))\right) / \mathrm{Aut}(\Gamma)\right]$

$$\overline{\mathcal{M}}_{g,n}(X,\beta) = \bigsqcup_{\Gamma \in G_{g,n,\beta}} \mathcal{M}_\Gamma$$ disjoint union of finitely many strata

$v \in V(\Gamma) \longleftrightarrow C_v$ connected component of $\tilde{C}$

$e \in E(\Gamma) \longleftrightarrow$ node y_e of C

$g(v) = \mathrm{genus}(C_v) \in \mathbb{Z}_{\geq 0}$, $\beta(v) = u_*[C_v] \in H_2(X;\mathbb{Z})$

If $\beta(v)=0$ then $2g(v)-2+n(v)>0$.

<u>Example</u> $\overline{\mathcal{M}}_{0,2}$ is empty $\quad \dim \mathcal{M}_{0,2}^{pre} = -1$

top stratum $\quad \Gamma = 1 - \bullet - 2 \quad \mathcal{M}_\Gamma = [\cdot/\mathbb{C}^*] \quad \dim \mathcal{M}_\Gamma = -1$

$$\mathrm{aut}(x_1 \; x_2) = \mathbb{C}^*$$

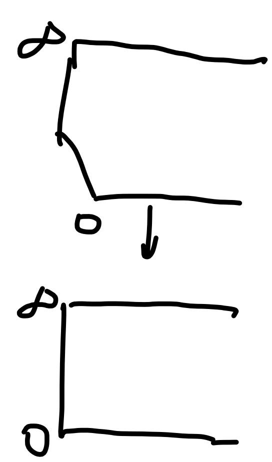
$$\#V(\Gamma) = 1, \quad \#E(\Gamma) = 0$$

$$\Gamma = 1 - \bullet - \bullet \cdots \bullet - 2$$

$$\#V(\Gamma) = m, \quad \#E(\Gamma) = m-1$$

$$\mathrm{aut}(x_1 \; \cdots \; x_2) = (\mathbb{C}^*)^m$$

$$\mathcal{M}_\Gamma = [\cdot/(\mathbb{C}^*)^m] \quad \dim \mathcal{M}_\Gamma = -m$$



$$\left[Bl_{(0,0)}(\mathbb{A}^1 \times \mathbb{P}^1) \big/ (\mathbb{C}^*)^2 \right]$$

$$\mathbb{A}^1 \times \mathbb{P}^1$$

$\mathbb{A}^1$

$$\left[\mathbb{A}^1 / (\mathbb{C}^*)^2 \right] = \underbrace{\left[0 / (\mathbb{C}^*)^2 \right]}_{\mathcal{M}_{1-\bullet-\bullet-2}} \cup \underbrace{\left[1 / \mathbb{C}^* \right]}_{\mathcal{M}_{1-\bullet-2}}$$

<u>Example</u> $\quad H_2(\mathbb{P}^1;\mathbb{Z}) = \mathbb{Z}\ell \qquad \ell = [\mathbb{P}^1]$

<u>Notation</u> $\quad \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) := \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d\ell)$

$\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 1)$ is a point

represented by $id : \mathbb{P}^1 \to \mathbb{P}^1$

$$\begin{tikzcd} \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1, 1) \arrow[r, "ev_1"] \arrow[d] & \mathbb{P}^1 \\ \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 1) & \end{tikzcd}$$

$$\mathbb{P}^1 \cong \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1, 1) \xrightarrow{ev_1} \mathbb{P}^1$$

$$p \in \mathbb{P}^1 \mapsto [id : (\mathbb{P}^1, p) \to \mathbb{P}^1] \mapsto p$$

$$\begin{tikzcd} (p,p) & \overline{\mathcal{M}}_{0,2}(\mathbb{P}^1, 1) \cong \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "ev_2 = pr_2"] \arrow[d, "\pi = pr_1"'] \arrow[dr, "ev_1"] & \mathbb{P}^1 \\ p \arrow[u, mapsto] & \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1, 1) \arrow[u, bend left, "\sigma_1"] \arrow[r, "ev_1", "\cong"'] & \mathbb{P}^1 \end{tikzcd}$$

§1.2 Perfect Obstruction Theory and the Virtual Fundamental Class

$\overline{\mathcal{M}}_{g,n}(X,\beta)$ is usually singular, so it does not have a tangent bundle in general, but it has a virtual tangent bundle which is a two term complex of vector bundles

$$\mathbb{E}^\vee = [E_0 \xrightarrow{\phi} E_1] = T^{vir} \quad \text{virtual tangent bundle}$$

$$0 \to \underset{\underset{\ker(\phi)}{\shortparallel}}{T^1} \to E_0 \xrightarrow{\phi} E_1 \to \underset{\underset{\mathrm{Coker}(\phi)}{\shortparallel}}{T^2} \to 0$$

At $\xi = [(C,\underline{x}), u] \in \overline{\mathcal{M}}_{g,n}(X,\beta)$, we have the following exact sequence of cpx vector spaces

$$0 \to \mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(x_1+\cdots+x_n), \mathcal{O}_C) \to H^0(C, u^*TX) \to T^1_\xi$$

$$\to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(x_1+\cdots+x_n), \mathcal{O}_C) \to H^1(C, u^*TX) \to T^2_\xi \to 0$$

- $\mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(x_1+\cdots+x_n), \mathcal{O}_C) = \mathrm{aut}(C,\overline{x}) := T_{\mathrm{id}_C} \mathrm{Aut}(C,\overline{x})$

  space of infinitesimal automorphisms of the domain/worldsheet $(C,\underline{x})$. $(C,\underline{x})$ is stable $\Leftrightarrow$ $\mathrm{aut}(C,\underline{x}) = 0$

If C is smooth then $\mathrm{aut}(C,\underline{x}) = H^0(C, T_C(-x_1 \cdots -x_n))$
space of holomorphic vector fields on C which vanish at $x_1, \dots, x_n$

- $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(x_1 + \cdots + x_n), \mathcal{O}_C) = \mathrm{def}(C, \underline{x})$
  space of infinitesimal deformation of $(C, \underline{x})$
- $H^0(C, u^* T_X) = \mathrm{def}(u)$ infinitesimal deformations of the map u for a fixed domain $(C, \underline{x})$
- $H^1(C, u^* T_X) = \mathrm{obs}(u)$ obstructions to deforming the map u for a fixed domain $(C, \underline{x})$

T^1_ξ tangent space at ξ

T^2_ξ obstruction

$$\mathcal{X} = \overline{\mathcal{M}}_{g,n}(X, \beta)$$

target $\downarrow$ virtually smooth of relative $\dim_{\mathbb{C}}$ $d^{\mathrm{vir}}_{\mathcal{X}/\mathcal{M}}$

$$\mathcal{M} = \mathcal{M}^{\mathrm{pre}}_{g,n} \quad \text{smooth of } \dim_{\mathbb{C}} 3g-3+n$$
$$= \dim \mathrm{def}(C, \underline{x}) - \dim \mathrm{aut}(C, \underline{x})$$

$$d^{\mathrm{vir}}_{\mathcal{X}/\mathcal{M}} = h^0(C, u^* T_X) - h^1(C, u^* T_X)$$

Riemann-Roch $\leftarrow$ M. Bertola's course

$$= \deg(u^*T_X) + \operatorname{rank}(u^*T_X)(1-g)$$

$$= \int_\beta c_1(T_X) + (\dim_{\mathbb{C}} X)(1-g)$$

$$d^X_{g,n,\beta} := \text{virtual dim of } \overline{\mathcal{M}}_{g,n}(X,\beta) := \operatorname{rank} T^{vir}$$

$$= \underset{3g-3+n}{\dim \mathcal{M}} + d^{vir}_{\mathcal{X}/\mathcal{M}}$$

$$= \int_\beta c_1(T_X) + (\dim_{\mathbb{C}} X - 3)(1-g) + n$$

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \in H_{2d^{vir}_{g,n,\beta}}(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q})$$

virtual fundamental class

<u>Definition</u> Let X be a nonsingular projective variety. We say X is convex if for any genus-$\underline{0}$ stable map $u:(C,x_1,\dots x_n)\to X$, $H^1(C,u^*TX)=0$

<u>Examples</u> projective spaces $\mathbb{P}^m = Gr(1,m+1)$

Grassmannian $Gr(k,m)$

generalized flag varieties G/P

<u>Proposition</u> If X is convex then for any n, β the obstruction sheaf $\mathcal{T}=0$ on $\overline{\mathcal{M}}_{0,n}(X,\beta)$.
$\overline{\mathcal{M}}_{0,n}(X,\beta)$ is a proper smooth DM stack of dim $d^X_{0,n,\beta}$
compact complex orbifold

$\mathcal{T} = \mathcal{T}^1$ vector bundle of rank $d^X_{0,n,\beta}$

$$[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{vir} = [\overline{\mathcal{M}}_{0,n}(X,\beta)] \text{ fundamental class}$$

<u>Example</u> (constant stable maps)

$\overline{\mathcal{M}}_{g,n}(X,0) \cong \overline{\mathcal{M}}_{g,n} \times X$ proper smooth DM stack of dim $3g-3+n+N$, $N=\dim_{\mathbb{C}} X$

$[u:(C,\underline{x})\to X]$ $\quad ([(C,\underline{x})], p) = \xi$

$u(z)=p \ \forall z\in C$

$$0 \to aut(C,\underline{x})=0 \to H^0(C,u^*T_X)=T_pX \to \mathcal{T}^1_\xi$$
$$\to def(C,\underline{x}) = T_{(C,\underline{x})}\overline{\mathcal{M}}_{g,n} \to H^1(C,u^*TX) \to \mathcal{T}^2_\xi$$
$$H^1(C,\mathcal{O}_C)\otimes T_pX = H^\vee_{[(C,\underline{x})]}\otimes T_pX$$

$$\mathcal{T}^{vir} = \mathcal{T}^1 - \mathcal{T}^2$$

$\mathcal{T}^1 = \mathcal{T}_{\overline{\mathcal{M}}_{g,n}\times X}$ vector bundle of rank $3g-3+n+N$

$\mathcal{T}^2 = \mathcal{H}^\vee \boxtimes TX$ $\qquad gN$

$$\Big[\overset{\mathcal{X}\,=}{\overline{\mathcal{M}}_{g,n}(X,0)}\Big]^{vir} = \underbrace{e(\mathcal{H}^{\vee}\boxtimes TX)}_{\in\, H^{2gN}(\mathcal{X};\mathbb{Q})} \cap \underbrace{[\overline{\mathcal{M}}_{g,n}\times X]}_{=\,[\overline{\mathcal{M}}_{g,n}]\times[X]\;\in\; H_{2(3g-3+n+N)}(\mathcal{X};\mathbb{Q})}$$

$$[\overline{\mathcal{M}}_{g,n}(X,0)]^{vir} \in H_{2((N-3)(1-g)+n)}(\mathcal{X};\mathbb{Q})$$

§1.3 GW invariants of Calabi-Yau 3-folds

$$d^X_{g,n,\beta} = \int_\beta c_1(TX) + (\underset{\underset{\dim X}{\parallel}}{N}-3)(1-g)+n$$

If X is a CY 3-fold then $c_1(TX)=0$, $N-3=0$

$\forall g\in\mathbb{Z}_{\geq 0}$, $\beta\in H_2(X;\mathbb{Z})$

$$[\overline{\mathcal{M}}_{g,0}(X,\beta)]^{vir} \in H_0(\overline{\mathcal{M}}_{g,0}(X,\beta);\mathbb{Q})$$

$$N^X_{g,\beta} := \int_{[\overline{\mathcal{M}}_{g,0}(X,\beta)]^{vir}} 1 \in \mathbb{Q}$$

genus g, degree β GW invariant of X

$g\geq 2$

$$F^X_g(Q) := \sum_\beta N^X_{g,\beta}\, Q^\beta$$

A-model genus g free energy

<u>Remark</u> If X is non-compact, $\overline{M}_{g,n}(X,\beta)$ is usually non-compact, but it is still a DM stack with a perfect obstruction theory of virtual dim $d^X_{g,n,\beta}$.

If X is a noncompact CY 3fold but $\overline{M}_{g,0}(X,\beta)$ happens to be compact for some g,β then

$$N_{g,\beta} = \int_{[\overline{M}_{g,0}(X,\beta)]^{vir}} 1 \in \mathbb{Q} \quad \text{is defined.}$$

i_0 ← zero section

<u>Example 1</u> $X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathbb{P}^1$

$$H_2(X;\mathbb{Z}) = H_2(\mathbb{P}^1;\mathbb{Z}) = \mathbb{Z}\ell$$

$$i_0 : \overline{M}_{g,0}(\mathbb{P}^1,d) \hookrightarrow \overline{M}_{g,0}(X,d)$$

is an isomorphism if $d \neq 0$

(empty if $d<0$)

$N^X_{g,d} \in \mathbb{Q}$ defined for all $d \in \mathbb{Z}_{>0}$

<u>Example 2</u> $X = K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{\quad} \mathbb{P}^2$

$$H_2(X;\mathbb{Z}) = H_2(\mathbb{P}^2;\mathbb{Z}) = \mathbb{Z}\ell$$

$i_0 : \overline{M}_{g,0}(\mathbb{P}^2,d) \hookrightarrow \overline{M}_{g,0}(X,d)$ is an isomorphism if $d \neq 0$ (empty if $d>0$)

$N^X_{g,d} \in \mathbb{Q}$ defined for all $d \in \mathbb{Z}_{>0}$

Example 1, 2 are toric CY 3folds

A CY 3fold X is toric if $(\mathbb{C}^*)^3 \overset{\text{open}}{\underset{\text{dense}}{\subset}} X$

The action of $(\mathbb{C}^*)^3$ on itself extends to X.

- The Topological Vertex provides a closed formula of
  $F^X_\beta(\lambda) = \sum_g N^X_{g,\beta} \lambda^{2g-2}$

- The Remodeling Conjecture provides an algorithm to compute $F^X_g(Q)$

§1.4 Descendant and ancestor GW invariants

X nonsingular projective variety $/\mathbb{C}$

evaluation maps: $ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$

$ev_i^* : H^*(X;\mathbb{Q}) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$

universal curve — universal map

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n+1}(X,\beta) \arrow[r, "ev_{n+1}"] \arrow[d, "\pi"] & X \\ \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[u, bend left, "\sigma_i"] & \end{tikzcd}$$

ω_π relative dualizing sheaf

$$\mathbb{L}_i = \sigma_i^* \omega_\pi \qquad T^*_{x_i} C$$

$$\begin{array}{c} \downarrow \\ \overline{\mathcal{M}}_{g,n}(X,\beta) \qquad [(C,\underline{x}),u] \end{array}$$

ψ-classes $\quad \psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q}) \qquad i=1,\dots,n$

$$\begin{tikzcd} \mathcal{H} = \pi_* \omega_\pi \arrow[d] & H^0(C,\omega_C) \arrow[l, phantom, "\supset"] \arrow[d] \\ \overline{\mathcal{M}}_{g,n}(X,\beta) & {[(C,\underline{x}),u]} \arrow[l, phantom, "\ni"] \end{tikzcd}$$

if C is smooth then $\omega_C = \Omega^1_C$

$\mathcal{H}$ Hodge bundle

vector bundle of rank g

λ-classes $\quad \lambda_j = c_j(\mathcal{H}) \in H^{2j}(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q}) \qquad j=1,\dots,g$

Remark

$$\begin{tikzcd}
\mathfrak{X} \arrow[d] & \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[d, "\mathrm{forget}"] \arrow[dd, bend left, "p"] & {[(C,\underline{x}),u]} \arrow[d, mapsto] \\
\mathcal{M} \arrow[d] & \mathcal{M}^{\mathrm{pre}}_{g,n} \arrow[d, "\mathrm{stab}"] & (C,\underline{x}) \arrow[d, mapsto] \\
\overline{\mathcal{M}} & \overline{\mathcal{M}}_{g,n} & (C,\underline{x})^{\mathrm{stab}}
\end{tikzcd}$$

$$\psi_i = (\mathrm{forget})^* \psi_i \neq p^* \psi_i =: \bar{\psi}_i$$

(gravitational) descendants

ancestor

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} : H^*(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q}) \longrightarrow H^{*-2d^X_{g,n,\beta}}(\underset{\text{point}}{\bullet};\mathbb{Q})$$

$$H^*(\bullet;\mathbb{Q}) = H^0(\bullet;\mathbb{Q}) = \mathbb{Q}$$

$\gamma_1, \dots, \gamma_n \in H^*(X;\mathbb{Q})$, $a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}$

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle^X_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^n \left(\psi_i^{a_i} \, ev_i^* \gamma_i\right)$$

for simplicity, assume $\gamma_i \in H^{2k_i}(X;\mathbb{Q})$

$= 0$ unless $\sum_{i=1}^n (a_i + k_i) = d^X_{g,n,\beta}$

(genus g deg β) descendant GW invariants of X

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle^X_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n \left(\psi_i^{a_i} ev_i^* \gamma_i \right) \in \mathbb{Q}$$

ancestor GW invariants of X

$$= \int_{[\overline{\mathcal{M}}_{g,n}]} \Omega^X_{g,n,\beta}(\gamma_1 \cdots \gamma_n) \prod_{i=1}^n \psi_i^{a_i}$$

$a_1 = \cdots = a_n = 0$ $\quad \langle \gamma_1 \cdots \gamma_n \rangle^X_{g,n,\beta} = \langle \tau_0(\gamma_1) \cdots \tau_0(\gamma_n) \rangle^X_{g,n,\beta}$ primary GW invariants of X

$$\Omega^X_{g,n,\beta} : H^*(X;\mathbb{Q})^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$$

$$p_*^{vir} : H^k(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q}) \longrightarrow H^{k+2(3g-3+n-d^X_{g,n,\beta})}$$

$$p_*^{vir}(\alpha) = p_*([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \cap \alpha)$$

$$\in H_{2d^X_{g,n,\beta}-k}(\overline{\mathcal{M}}_{g,n};\mathbb{Q}) \xrightarrow{\cong} H^{2(3g-3+n-d^X_{g,n,\beta})+k}(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$$

Poincaré duality

projection formula $\forall \alpha \in H^*(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q})$ $\forall \gamma \in H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$

$$p_*^{vir}(\alpha \cup p^*\gamma) = (p_*^{vir}\alpha) \cup \gamma \in H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$$

$$\Omega^X_{g,n,\beta}(\gamma_1 - \gamma_n) = p_*^{vir}\left(\prod_{i=1}^n ev_i^* \gamma_i \right)$$

Novikov ring

$$\Omega^X_{g,n} := \sum_{\beta \in \mathrm{Eff}} Q^\beta \Omega^X_{g,n,\beta} : H^*(X;\Lambda)^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n};\Lambda)$$

§1.5 CohFT Axioms

Theorem (Li-Tian, Behrend)

$\Omega^X = (\Omega^X_{g,n})_{2g-2+n>0}$ is a CohFT on

$(H^*(X,\Lambda), \eta)$ where $\eta(\alpha,\beta) = \int_X \alpha \cup \beta$

Poincaré pairing

D. Lewanski's talk

i) (Symmetry) $\Omega^X_{g,n}$ is S_n-invariant

$[\overline{M}_{g,n}(X,\beta)]^{vir}$ is S_n-invariant

ii) (Gluing)

$\rho: \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$

$\rho^* \Omega_{g,n}(\gamma_1, \dots \gamma_n) = \eta^{\alpha\beta}\, \Omega(e_\alpha, e_\beta, \gamma_1 \cdots \gamma_n)$

$$\begin{tikzcd} Z_1 \arrow[r] \arrow[d] & \overline{M}_{g-1,n+2}(X,\beta) \arrow[d, "ev_1 \times ev_2"] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd} \qquad \begin{tikzcd} Z_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g,n}(X,\beta) \arrow[d, "p"] \\ \overline{M}_{g-1,n+2} \arrow[r, "\rho"] & \overline{M}_{g,n} \end{tikzcd}$$

$\Phi: Z_1 \to Z_2$ $\quad \Phi_* \Delta^! \overline{M}_{g-1,n+2}(X,\beta) = \rho^! [\overline{M}_{g,n}(X,\beta)]^{vir}$

$$\sigma : \overline{M}_{g_1, 1+|I_1|} \times \overline{M}_{g_2, 1+|I_2|} \to \overline{M}_{g,n} \qquad g_1+g_2=g, \quad I_1 \sqcup I_2 = \{1,\dots,n\}.$$

$\beta_1+\beta_2=\beta$

$$\begin{tikzcd} Z_{\beta_1,\beta_2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g_1,1+|I_1|}(X,\beta_1)\times \overline{M}_{g_2,1+|I_2|}(X,\beta_2) \arrow[d, "ev_1\times ev_1"] \\ X \arrow[r, "\Delta"] & X\times X \end{tikzcd} \qquad \begin{tikzcd} W_\beta \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g,n}(X,\beta) \arrow[d, "p"] \\ \overline{M}_{g_1,1+|I_1|}\times\overline{M}_{g_2,1+|I_2|} \arrow[r, "\sigma"] & \overline{M}_{g,n} \end{tikzcd}$$

$$\Phi : \bigsqcup_{\beta_1+\beta_2=\beta} Z_{\beta_1,\beta_2} \to W_\beta$$

$$\Phi_*\left(\sum_{\beta_1+\beta_2=\beta} \Delta^!\left([\overline{M}_{g_1,1+|I_1|}(X,\beta_1)]^{vir}\times[\overline{M}_{g_2,1+|I_2|}(X,\beta_2)]^{vir}\right)\right)$$
$$= \sigma^! \, [\overline{M}_{g,n}(X,\beta)]^{vir}$$

iii) (unit) $\quad \pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$

$$\pi^*\Omega_{g,n} = \Omega_{g,n}$$
$$\Omega_{0,3}(\gamma_1,\gamma_2,1) = \eta(\gamma_1,\gamma_2)$$

$\pi : \overline{M}_{g,n+1}(X,\beta) \to \overline{M}_{g,n}(X,\beta)$ flat morphism of relative dim 1

$\pi^*[M_{g,n}(X,\beta)]^{vir} = [\overline{M}_{g,n+1}(X,\beta)]^{vir} \Rightarrow \Omega_{0,3,\beta}(\gamma_1,\gamma_2,1)=0$ if $\beta\neq 0$

Example on p.8 $\Rightarrow$ $\Omega_{0,3,0}(\gamma_1,\gamma_2,\gamma_3) = \int_X \gamma_1\gamma_2\gamma_3$.

§1.6 String, dilaton, and divisor equations

Suppose (g,n,β) is stable i.e. $\beta\neq 0$ or $2g-2+n>0$.

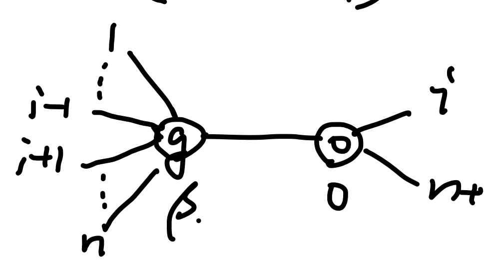
$$\overline{\mathcal{M}}_{g,n+1}(X,\beta) \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}(X,\beta)$$

$$\mathbb{L}_i \quad i=1,\dots,n+1 \qquad\qquad \mathbb{L}_i' \quad i=1,\dots,n$$

Comparison Lemma

(a) $i=1,\dots,n \qquad \mathbb{L}_i = \pi^*\mathbb{L}_i' \otimes \mathcal{O}(D_{i,n+1})$

($\Rightarrow$ geometric string)



(b) $\mathbb{L}_{n+1} = \omega_\pi \otimes \mathcal{O}\left(\sum_{i=1}^n D_{i,n+1}\right)$ ($\Rightarrow$ geometric dilaton)

Recall that (c) $\pi^*[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} = [\overline{\mathcal{M}}_{g,n+1}(X,\beta)]^{vir}$

(a), (b), (c) imply the following theorem:

Theorem pose that (g,n,β) is stable

(1) (string equation)

$$\langle \tau_{a_1}(\gamma_1)\cdots\tau_{a_n}(\gamma_n)\,\tau_0(1)\rangle^X_{g,n+1,\beta}$$

$$= \sum_{i=1}^n \langle \tau_{a_1}(\gamma_1)\cdots\tau_{a_i-1}(\gamma_i)\cdots\tau_{a_n}(\gamma_n)\rangle^X_{g,n,\beta}$$

(2) (dilaton equation)

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n)\, \tau_1(1) \rangle^X_{g,n+1,\beta} = (2g-2+n) \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle^X_{g,n,\beta}$$

(3) $\gamma \in H^2(X;\mathbb{Q})$

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n)\, \tau_0(\gamma) \rangle^X_{g,n+1,\beta} = \int_\beta \gamma \, \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle^X_{g,n,\beta}$$
$$+ \sum_{i=1}^{n} \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_i-1}(\gamma_i \cup \gamma) \cdots \tau_{a_n}(\gamma_n) \rangle^X_{g,n,\beta}$$

<u>Corollary</u> Suppose that (g,n,β) is stable

(1)' (primary string equation)

$$\langle \gamma_1 \cdots \gamma_n\, 1 \rangle^X_{g,n+1,\beta} = 0$$

(3)' (primary divisor equation) $\gamma \in H^2(X;\mathbb{Q})$

$$\langle \gamma_1 \cdots \gamma_n\, \gamma \rangle^X_{g,n+1,\beta} = \int_\beta \gamma \, \langle \gamma_1 \cdots \gamma_n \rangle^X_{g,n,\beta}$$

§2 The Topological Vertex

Toric Calabi-Yau 3-folds

A Calabi-Yau 3-fold X is toric if it contains $T=(\mathbb{C}^*)^3$ as an open dense subset, and the action of $(\mathbb{C}^*)^3$ on itself extends to X.

Assume $X^0 = X^T$ is non-empty

$\forall p \in X^T$ $\qquad T' \subset T$

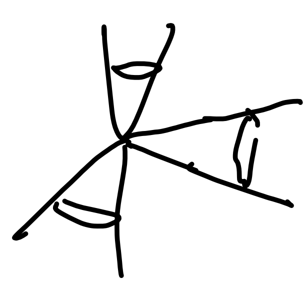
$\qquad\qquad\qquad T' \cong (\mathbb{C}^*)^2$ CY torus acts trivially on $\Lambda^3_p TX$

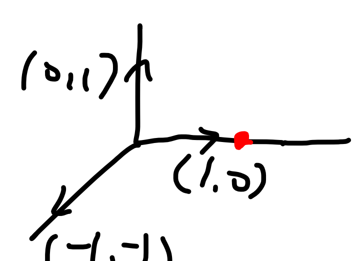
X^1 = union of 0-dim'l and 1-dim'l T-orbit

1-skeleton

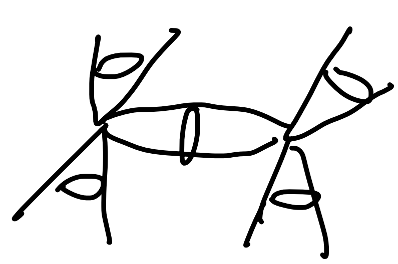
$X \qquad\qquad X^1 \qquad\qquad \Gamma_X$

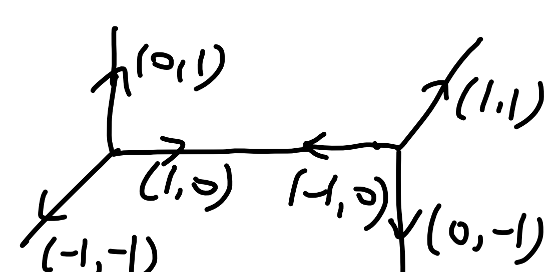
$\mathbb{C}^3$



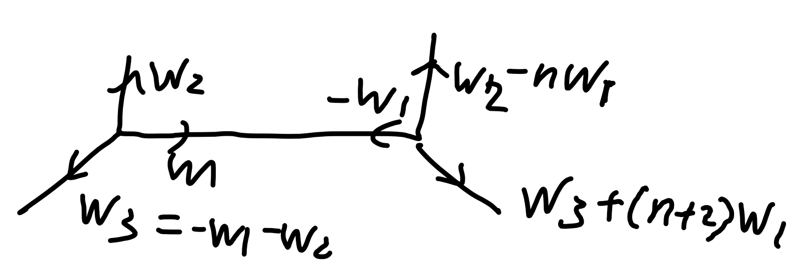


$\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

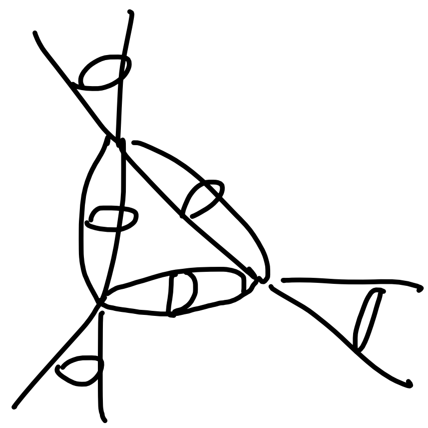


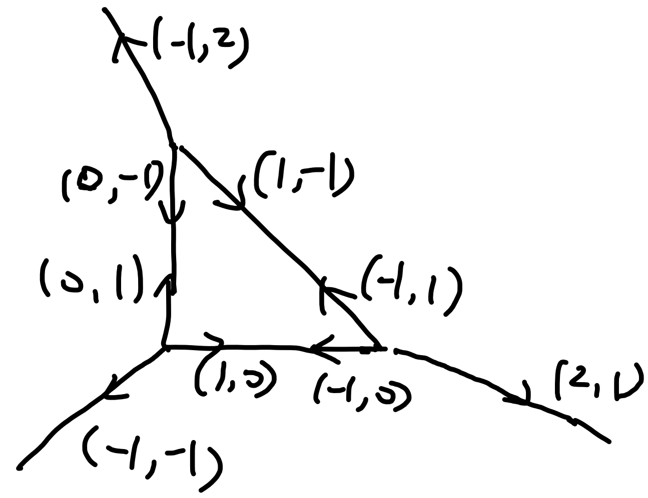


$\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n-2)$



$\mathcal{O}_{\mathbb{P}^2}(-3)$



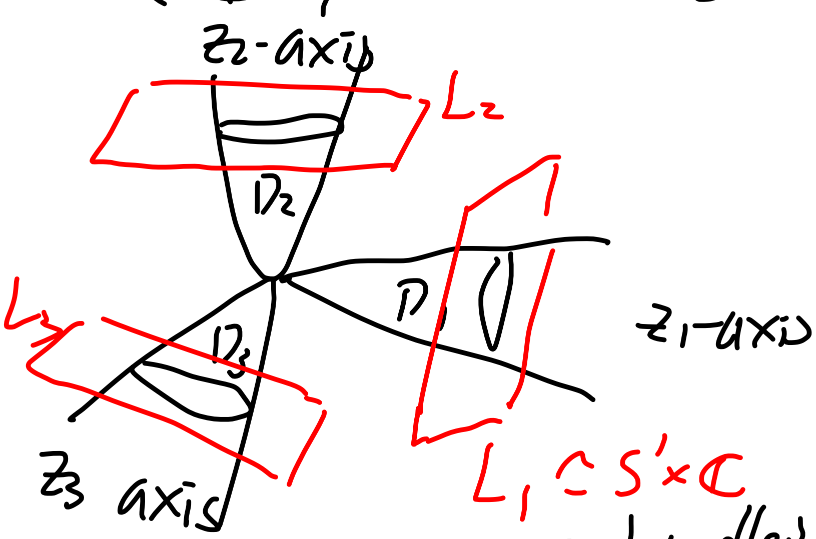


Algorithm of AKMV (based on the large N duality)

Aganagic-Klemm-Mariño-Vafa, "The topological Vertex" CMP

01. The Topological Vertex

There exists open GW invariants $F_{g,\vec{\mu}}(\vec{n})$ of $(\mathbb{C}^3, L_1\cup L_2\cup L_3)$ where $g\in\mathbb{Z}_{\geq 0}$



$\vec{\mu}=(\mu^1,\mu^2,\mu^3)$, $\vec{n}=(n_1,n_2,n_3)$

$\mu^i=(\mu^i_1,\dots,\mu^i_{\ell_i})$ winding #'s

$n_i\in\mathbb{Z}$ framing of L_i

g handles and $(\ell_1+\ell_2+\ell_3)$ holes

$$u:\Big(\Sigma,\ \partial\Sigma=\coprod_{i=1}^{3}\coprod_{j=1}^{\ell_i}R_{ij}\Big)\xrightarrow{hol.}(\mathbb{C}^3, L_1\cup L_2\cup L_3)$$

$u(R_{ij})\subset L_i$, $\quad u_*[R_{ij}]=\mu^i_j[\partial D_i]$ $\quad \mu^i_j\in\mathbb{Z}_{>0}$

$C_{\vec{\mu}}(\lambda, \vec{n})$ generating function of $F^{\bullet}_{\chi, \vec{\mu}}(\vec{n})$
disconnected version of $F_{g, \vec{\mu}}(\vec{n})$

02. Gluing algorithm GW invariants of any toric CY 3-folds can be expressed in terms of $C_{\vec{\mu}}(\lambda, \vec{n})$

03. Closed formula
$$C_{\vec{\mu}}(\lambda; \vec{n}) = q^{(\sum_{i=1}^{3} \kappa_{\mu^i} n_i)/2} \mathcal{W}_{\vec{\mu}}(q)$$
where $q = e^{\sqrt{-1}\lambda}$, $\kappa_\mu = \sum \mu_i(\mu_i - 2i + 1)$

$\mathcal{W}_{\vec{\mu}}(q)$ is related to the colored HOMFLY polynomial of a link with 3 component

<u>Mathematical Theory of LLLZ</u> (based on relative GW theory and degeneration formula)

J. Li, –, K. Liu, J. Zhou, "A mathematical theory of the topological vertex" G&T

R1. We define formal relative GW invariants of relative formal toric Calabi-Yau (FTCY) 3-folds.
These invariants are refinement and generalization of GW invariants of smooth toric CY 3folds

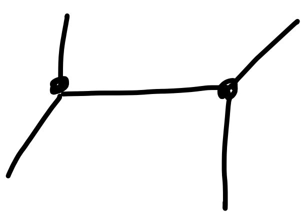
R2. Formal relative GW invariants satisfy the degeneration formula. In particular, they can be expressed in terms of $\widehat{C}_{\vec{\mu}}(\lambda, \vec{n})$ formal relative GW invariants of an indecomposable relative FTCY 3-fold

R3. $\widehat{C}_{\vec{\mu}}(\lambda; \vec{n}) = q^{(\sum_{i=1}^{3} \kappa_{\mu^i} n_i)/2}\, \widetilde{\mathcal{W}}_{\vec{\mu}}(q)$

where $\widetilde{\mathcal{W}}_{\vec{\mu}}(q)$ is a combinatoric expression in terms of representations of symmetric groups.

<u>Remark</u>

Norman Do and Brett Parker, "The tropological vertex" (relative GW invariants of log CY, enumeration of tropical curves) planar trivalent graph Γ_X → T-equivariant tubular/formal neighborhood of X' in X → GW invariants

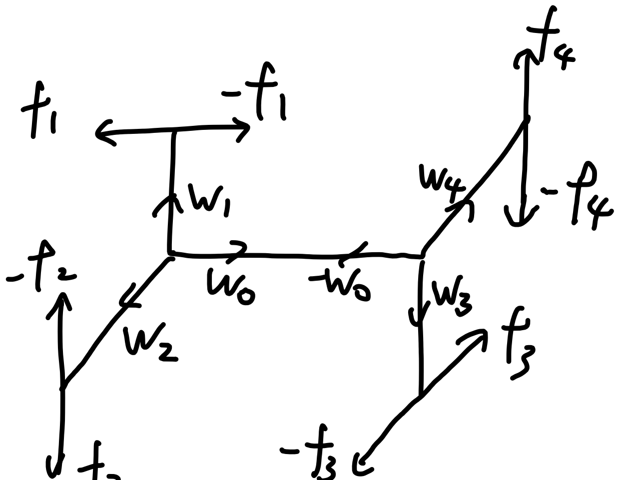


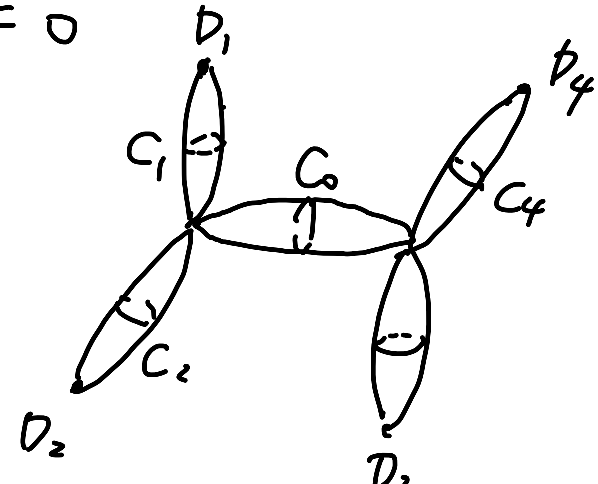
$$N^X_{g,\beta} \overset{``}{=}{}^{"} \int_{[\overline{M}_{g,0}(X,\beta)]^{vir}} 1 = \sum_{\Gamma\in G^T_{g,\beta}} \int_{[F_\Gamma]^{vir}} \frac{1}{e_{T'}(N^{vir}_\Gamma)}$$

$$\overline{M}_{g,0}(X,\beta)^T = \bigsqcup_{\Gamma\in G^T_{g,\beta}} F_\Gamma$$

Formal Toric Calabi-Yau (FCTY) graphs

FTCY graph Γ → relative (log) FTCY 3-fold $Y^{rel}_\Gamma = (\hat{Y}, \hat{D})$, $K_{\hat{Y}} + \hat{D} = 0$ → formal relative GW invariants $F^\Gamma_{g,\vec{d},\vec{\mu}}$





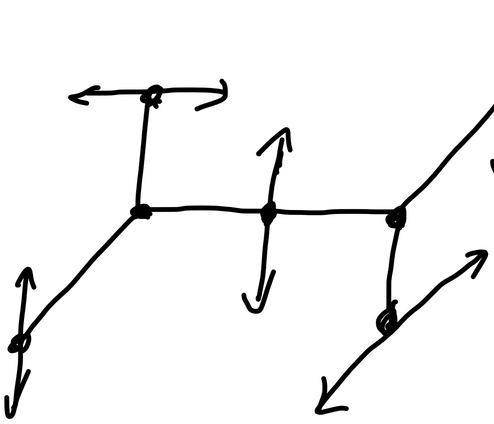
$f_1 = w_2 - n_1 w_1$ $\quad f_3 = w_4 - n_3 w_3$

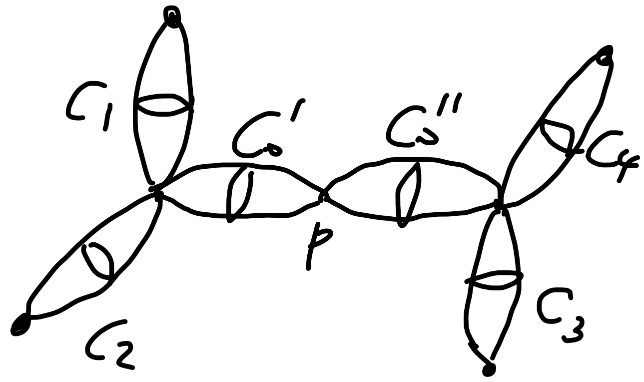
$f_2 = w_0 - n_2 w_2$ $\quad f_4 = -w_0 - n_4 w_4$

$$N_{C_i/\hat{Y}} = \mathcal{O}_{\mathbb{P}^1}(n_i) \oplus \mathcal{O}_{\mathbb{P}^1}(-n_i-2)$$

$i = 1, 2, 3, 4$

$$N_{C_0/\hat{Y}} = \mathcal{O}_{\mathbb{P}^1}(n_0) \oplus \mathcal{O}_{\mathbb{P}^1}(-n_0-2)$$



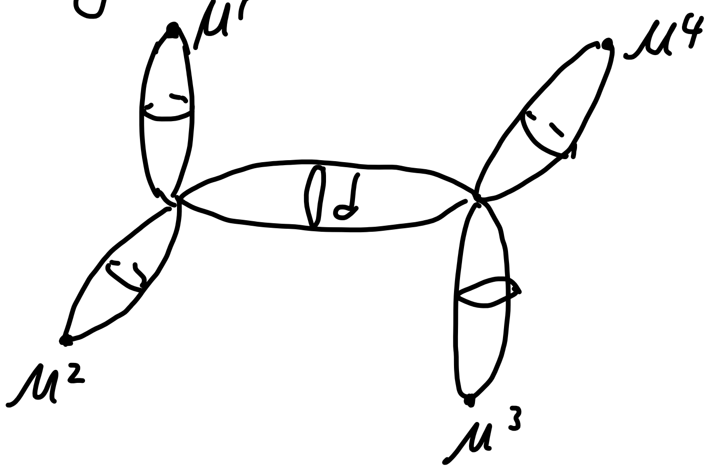


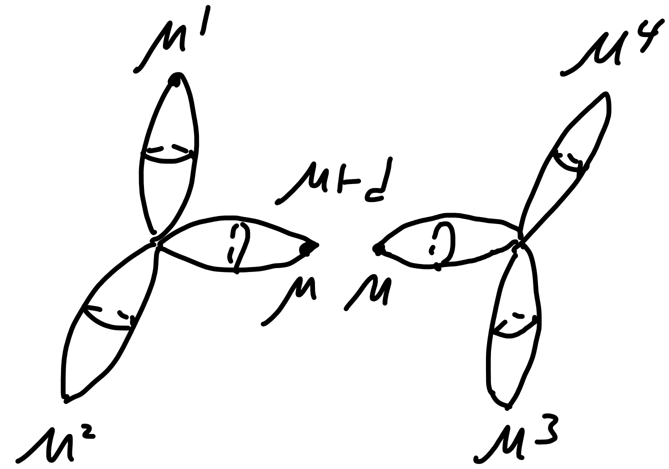
$$N_{C_0'/\hat{Y}'} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a-1)$$

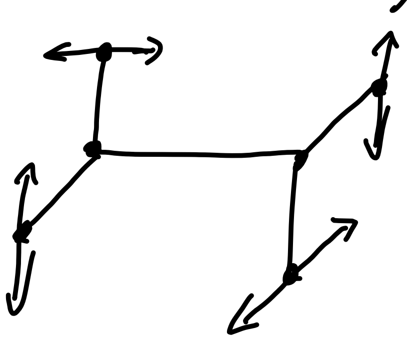
$$N_{C_0''/\hat{Y}'} = \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(-b-1)$$

$$a+b=n_0$$

Degeneration formula

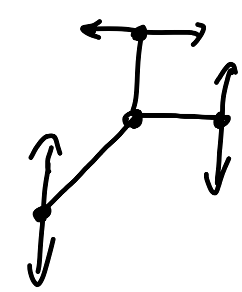


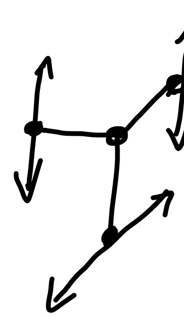




Γ_0

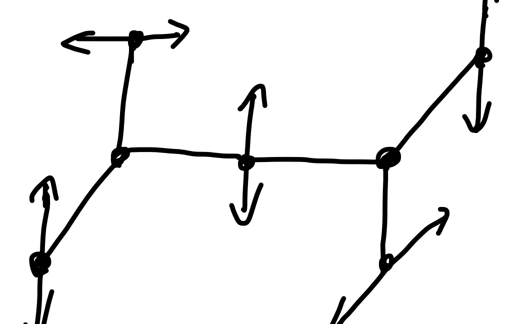
Γ_1

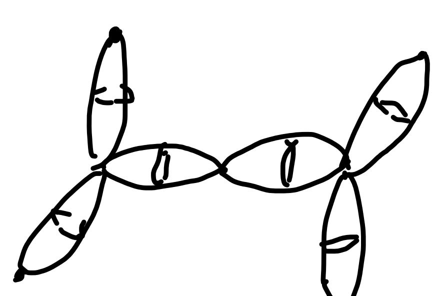




Γ_2

Γ

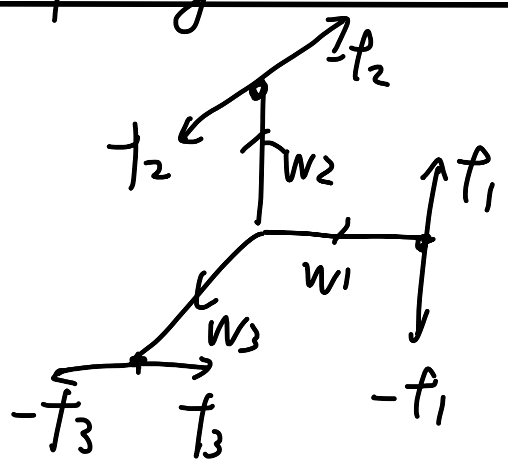




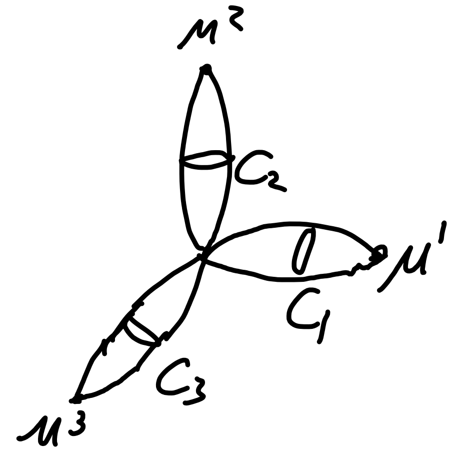
$$z_\nu = \mathrm{Aut}(\nu)\prod_j \nu_j$$

$$F^{\bullet\Gamma_0}_{\chi,d,\vec{\mu}} = \sum_{\substack{\chi_1,\chi_2,\nu\vdash d \\ \chi_1+\chi_2-2\ell(\nu)=\chi}} F^{\bullet\Gamma_1}_{\chi_1,(\mu^1,\mu^2,\nu)}\, z_\nu\, F^{\bullet\Gamma_2}_{\chi_2,(\mu^3,\mu^4,\nu)}$$

Topological Vertex



$\Gamma_{w_1, w_2, \vec{n}}$



$f_i = w_{i+1} - n_i w_i$

$N_{C_i/Y} = \mathcal{O}(n_i) \oplus \mathcal{O}(-n_i - 1)$

$$F^{\bullet}_{\chi, \vec{\mu}}(\vec{n}) := F^{\bullet\, \Gamma_{w_1, w_2, \vec{n}}}_{\chi, \vec{\mu}}$$

$\vec{\mu} = (\mu^1, \mu^2, \mu^3)$
$\vec{n} = (n_1, n_2, n_3)$

$$F^{\bullet}_{\vec{\mu}}(\lambda; \vec{n}) = \sum_{\chi} \lambda^{-\chi + l(\vec{\mu})} F^{\bullet}_{\chi, \vec{\mu}}(\vec{n})$$

$$\widetilde{F}^{\bullet}_{\vec{\mu}}(\lambda; \vec{n}) = (-1)^{\sum_{i=1}^{3} (n_i - 1)|\mu^i|} \sqrt{-1}^{\,l(\vec{n})} F^{\bullet}_{\vec{\mu}}(\lambda; \vec{n})$$

Define $\widetilde{C}_{\vec{\mu}}(\lambda; \vec{n}) = \sum_{|\nu^i| = |\mu^i|} \widetilde{F}^{\bullet}_{\vec{\nu}}(\lambda; \vec{n}) \prod_{i=1}^{3} \chi_{\mu^i}(\nu^i)$

Then $\widetilde{C}_{\vec{\mu}}(\lambda; \vec{n}) = q^{\sum \kappa_{\mu^i} n_i/2} \, \widetilde{C}_{\vec{\mu}}(\lambda; \vec{0}) \qquad q = e^{\sqrt{-1}\lambda}$

Recall that O3. $\Rightarrow$ $C_{\vec{\mu}}(\lambda; \vec{n}) = q^{\sum \kappa_{\mu^i} n_i/2} C_{\vec{\mu}}(\lambda; \vec{0})$

O3. (AKMV) $C_{\vec\mu}(\lambda;\vec 0) = W_{\vec\mu}(q) \xleftrightarrow{\text{ORV}}$ generating function of 3d partitions

K3. (LLLZ) $\hat C_{\vec\mu}(\lambda;\vec 0) = \tilde W_{\vec\mu}(q)$

It remains to show $W_{\vec\mu}(q) = \tilde W_{\vec\mu}(q)$

1-leg vertex (unknot) LLZ, OP = Okounkov-Pandharipande 2003

2-leg vertex (Hopf link) LLZ 2003

3-leg vertex : Toshio Nakatsu & Kanehisa Takasaki 2018

MOOP $\Rightarrow$

Maulik-Nekrasov-Okounkov-Pandharipande (MNOP)

Conjecture GW/DT correspondence for smooth projective 3-folds

GW/DT correspondence for toric CY 3folds

$\Longleftrightarrow$ topological vertex $\Updownarrow$ localization

GW vertex = DT vertex

gluing $\mathbb{C}^3$ charts

GW vertex : generating function of triple Hodge integrals

DT vertex : generating function of 3d partitions toric CY 3-orbifold : gluing $[\mathbb{C}^3/G]$ chart

G finite subgroup of $T' \cong (\mathbb{C}^*)^2$

orbifold DT vertex (Bryan-Cadman-Young)

generating function of colored 3d partitions

orbifold GW vertex (Ross)

generating function of triple (abelian) Hurwitz-Hodge integrals

GW/DT for $[\mathbb{C}^2/\mathbb{Z}_m] \times \mathbb{C}$ transverse A_{m-1}

Ross, Zong

orbifold GW/DT for general toric CY 3-orbifolds is not known.