Exercises XII

In the following exercises feel free to use the following facts.

Fact 1. Suppose that K is a field of characteristic zero and p a prime number. Let a be an element of K which is not a pth power in K. Then the polynomial $X^p - a$ is irreducible over K.

Fact 2. Let L/K be a finite extension of degree d. Consider the map $\det : L \to K$ which maps $\alpha \in L$ to the determinant of the K-linear map $L \to L$, $x \mapsto \alpha x$. You check easily that det (sometimes called the norm map) satisfies the following two properties:
(a) $\det(\alpha\beta) = \det(\alpha)\det(\beta)$, and
(b) $\det(\alpha) = \alpha^d$ if $\alpha \in K$.

1. Let K, $a \in K$, and p be as in Fact 1. Let $L = K(a^{1/p})$. Compute $\det(a^{1/p})$, where det is as in Fact 2. (Careful!)

2. Let K be a field of characteristic 0, and let p be an odd prime. Suppose that a is an element of K which is not a pth power in K. Show that $X^{p^2} - a$ is irreducible over K. Hint: show that $a^{1/p}$ is not a pth power in $K(a^{1/p})$.

3. Show by an example that the statement of exercise 2 is wrong if $p = 2$.

4. In this exercise K is a field with the following properties:
(a) The characteristic of K is 0,
(b) the algebraic closure $\bar{K}$ of K is finite over K, and
(c) $K \neq \bar{K}$.

Show that the finite group $G = Gal(\bar{K}/K)$ is a 2-group. Hint: Suppose G contains an element σ of odd prime order p in G. Show that the fixed field $\bar{K}^{<\sigma>}$ contains a primitive pth root of unity. Then use Kummer theory and Exercise 2.

Remark. This is the beginning of the proof that shows that fields as in Exercise 4 are special. Namely, it can only happen if $[\bar{K} : K] = 2$ and K is a "real closed field".

5. Let $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7$, etc be the prime numbers. Consider the group

$$\prod_{n \in \mathbf{N}} \mathbf{Z}/p_n\mathbf{Z}.$$

Show that any finite index subgroup is given by a sub-product

$$\prod_{n \in S} \mathbf{Z}/p_n\mathbf{Z}.$$

for some subset $S \subset \mathbf{N}$ containing all but finitely many $n \in \mathbf{N}$. (Feel free to use that the result holds for finite products of cyclic groups of pairwise distinct prime orders. But only if you actually understand what I mean by that...)

Warning. It isn't true that any subgroup of the product is a "sub-product".