Exercises VIII

1. A *derivation* of a ring R is a map $D : R \to R$ such that $D(x + y) = D(x) + D(y)$ and $D(xy) = xD(y) + D(x)y$. Note that the zero map $D(x) = 0$ is a derivation of any ring.

(a) Show that if $R = \mathbf{Z}$, $R = \mathbf{Q}$, or $R = \mathbf{Z}/n\mathbf{Z}$ then the only derivation of R is the zero derivation. (Hint: show that $D(1) = 0$ for any derivation of any ring.)

(b) Show that any derivation of $\mathbf{Z}[x]$ is of the form $D = g\mathrm{d}/\mathrm{d}x$ for some $g \in \mathbf{Z}[x]$.

(c) Let $f \in \mathbf{Q}[x]$ be a polynomial and let $R = \mathbf{Q}[x]/(f)$. For which f does R have a nonzero derivation?

2. Let $K = F(\alpha)/F$ be a primitive algebraic extension of fields of characteristic p. A *derivation of K over F* is a derivation D of K such that $D(a) = 0$ for all $a \in F$. We use the notation $\mathrm{Der}(K/F)$ to denote the set of derivations of K over F.

(a) When does K have a nonzero derivation D over F?

(b) Consider the subfield $F \subset L \subset K$ defined by the rule

$$L = \{x \in K \mid D(x) = 0, \forall D \in \mathrm{Der}(K/F)\}.$$

Characterize L in terms of α and $Irr(\alpha, F)$.

3. Prove that

$$\mathbf{Q} = \mathbf{Q}((2 + \frac{10}{9}\sqrt{3})^{1/3} + (2 - \frac{10}{9}\sqrt{3})^{1/3}).$$

4. Show that in a finite field every element is the sum of two squares.

5. Let F be a field of characteristic $p > 0$, and let $a \in F$. Show that $X^p - X + a$ either splits completely over F or is irreducible over F.

6. What is the Galois group of the polynomial $3x^6 - 12x^4 + 4x^3 + 12x^2 - 8$ over $\mathbf{Q}$? (This should be pretty tough! Feel free to use computer algebra to find the answer, but try to write up the arguments by hand. For example, write the roots as $\alpha_1, \ldots, \alpha_6$ and try to find expressions in them which are in $\mathbf{Q}$ to restrict the size of the Galois group.)