Honors Linear Algebra

Lecture XXI

Classification of Matrices.

§11.2 The rank theorem

Theorem Let $A \in \mathrm{Mat}(m \times n, \mathbb{F})$. Then there exist invertible matrices $P \in GL(n, \mathbb{F})$, $Q \in GL(m, \mathbb{F})$ s.t.

$$Q^{-1}AP = \begin{array}{c} \overset{\longleftarrow r \longrightarrow}{} \\ r\updownarrow \left(\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & 0 \\ 0 & & 1 & \\ \hline & 0 & & 0 \end{array}\right) \end{array}$$

where $r = \mathrm{rank}(A)$.

Equivalent form Given a linear map $f: V \to W$ where $\dim V = n$, $\dim W = m$ there exist bases $(v_1, \ldots, v_n)$ of V and $(w_1, \ldots, w_m)$ of W such that the matrix of f w.r.t. these bases is

$$\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ \hline & 0 & & 0 \end{array}\right).$$

Proof of matrix form.

There exist row operations

$$A \xrightarrow[\text{left mult by } E_1]{} \begin{pmatrix} \bullet & * & \cdots & * & * & \cdots & * \\ & & \bullet & * & \cdots & & \\ & & & & \bullet & \cdots & \end{pmatrix}$$

row echelon form.

Using more row operations (E_2) can make $\bullet \longrightarrow 1$.

Using column operations (right multiply by E_3) can get

$$\left(\begin{array}{c|c} \begin{matrix} 1 & * & & \\ & 1 & * & \\ & & \ddots & \\ & & & 1 \end{matrix} & * \\ \hline 0 & 0 \end{array}\right)$$

Using row operations (E_4) can get

$$\left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & * \\ \hline 0 & 0 \end{array}\right)$$

Using column operations (E_5) can get rid of $*$.

Thus

$$E_4E_2E_1AE_3E_5 = \left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right).$$

Proof of linear map case:

Pick basis ~~....~~ $v_{r+1}, \dots, v_n$ of $\ker(f)$

extend to $v_1, \dots, v_n$ basis of V

$\xrightarrow[\text{dim}^n \text{ formula for } f]{\text{see proof}}$ $f(v_1), \dots, f(v_r)$ basis of $\mathrm{Im}(f)$

set $w_i = f(v_i)$ $i = 1, \dots, r$

extend to basis $w_1, \dots, w_m$ of W.

Done.

§11.3. Jordan normal form.

Theorem (Goal for next couple of lectures)

MATRIX VERSION

~~..~~ Given an $n \times n$-matrix $A \in M(n \times n, \mathbb{C})$ over ~~with~~ complex ~~..~~ number there exists a $P \in GL(n, \mathbb{C})$ s.t.

$$P^{-1}AP = \begin{pmatrix} \begin{matrix} \lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1 \end{matrix} & & \\ & \boxed{} & \\ & & \ddots \end{pmatrix}.$$

Jordan blocks

Jordan normal form

<u>Equivalent form</u> Given an n-dimensional complex vector space V and $f: V \to V$ linear there exists a basis $(v_1, \ldots, v_n)$ of V such that the matrix of f w.r.t.

We will prove the theorem in its equivalent form.

Extra Nilpotent endomorphisms

Def^n Let V be a vecter space. An endo $f: V \to V$ is <u>nilpotent</u> iff there exists an $n \geq 1$ such that

$$f^n = f \circ \cdots \circ f \quad (n \text{ times})$$

is equal to 0.

Rmk. If a diagonal matrix is nilpotent then it is zero because

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^m = \begin{pmatrix} \lambda_1^m & & \\ & \ddots & \\ & & \lambda_n^m \end{pmatrix}$$

Thus ~~an~~ nilpotent endomorphism is diagonalizable iff it is zero.

Example $\dim V = 1$: only nilpotent operater is zero

$\dim V = 2$: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

<u>Lemma</u> Let V be a finite dimensional vector space. Let $f: V \to V$ be a nilpotent endo. There exists a basis $(v_1, \dots, v_n)$ of V such that the matrix of f w.r.t. $(v_1, \dots, v_n)$ looks like

$$\begin{pmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_k} \end{pmatrix}$$

with

$$J_i = \underbrace{\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}}_{n_i} \qquad n_i \geq 1, \quad \sum n_i = n$$

<u>Proof.</u> By induction on $n = \dim(V)$.

If $n = 0$ or $n = 1$ the lemma is clear (as $f = 0$).

Let m be the smallest integer ≥ 1 such that $f^m = 0$. Pick a vector $v_1 \in V$

such that $f^{m-1}(v_1) \neq 0$ but $f^m(v_1) = 0$ (of course).

Set
$$v_2 = f(v_1), \quad v_3 = f^2(v_1), \; \dots, \; v_m = f^{m-1}(v_1)$$

Set
$$U = \text{span}(v_1, v_2, \dots, v_m) = L(v_1, \dots, v_m).$$

Let $W \subset V$ be a subspace such that

(a) $W \cap U = 0$

(b) $f(W) \subset W$

(c) $\dim W$ maximal among W with (a) + (b).

~~We use~~ <u>Claim</u> $W + U = V$

The claim finishes the proof: By I.H. we can find a basis $w_1, \dots, w_{\dim(W)}$ s.t. the matrix of $f|_W : W \to W$ w.r.t. this basis looks as in the statement of the lemma.

Then $v_1, \dots, v_m, w_1, \dots, w_{\dim W}$ will be a basis for V (small detail omitted: you have to show $v_1, \dots, v_m$ are lin. indep., see exercises), and the matrix of f w.r.t. this will be

$$\left(\begin{array}{c|c} \begin{matrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{matrix} & \\ \hline & \begin{matrix} \text{matrix of} \\ f|_W \text{ w.r.t. } w_1, \dots, w_{\dim W} \end{matrix} \end{array}\right)$$

<u>Proof of Claim</u> Say $W + U \neq V$ to get a contradiction. Pick $v \in V$, $v \notin W+U$. Since $f^m(v) = 0 \in W+U$ we can replace v by $f^i(v)$ for some i and assume that $v \notin W+U$ but $f(v) \in W+U$. Write

$$f(v) = w + u$$

$$u = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m$$

If $\lambda_1 \neq 0$, then

$$f^m(v) = f^{m-1}(w+u)$$

$$= f^{m-1}(w) + f^{m-1}(\lambda_1 v_1 + \cdots)$$

$$= f^{m-1}(w) + \lambda_1 f^{m-1}(v_1) + \lambda_2 f^m(v_2) + \cdots$$

$$= f^{m-1}(w) + \lambda_1 f^{m-1}(v_1) = f^{m-1}(w) + \lambda_1 v_{m-1}$$

But $v_{m-1} \in U$ is nonzero and not in W by (a)

So $f^m(v) \neq 0$. Contradiction with $f^m = 0$.

Hence $\lambda_1 = 0$. Thus

$$u = \lambda_2 v_2 + \cdots + \lambda_m v_m = f(\lambda_2 v_1 + \cdots + \lambda_m v_{m-1})$$

Thus

$$f(v - \lambda_2 v_1 \cdots - \lambda_m v_{m-1}) = w \in W.$$

Then we conclude

$$W' = W + \mathbb{F} \cdot (v - \lambda_2 v_1 \dots - \lambda_m v_{m-1})$$

satisfies (a). But also $W' \cap U = 0$ because $v - \lambda_2 v_1 \dots - \lambda_m v_{m-1} \notin W + U$ (small detail omitted; see exercises).

Thus (b) holds. Then W' contradicts (c) for W and we see the claim is true

<u>End proof lemma</u>.