Honors Linear Algebra

Lecture XXII

Cayley - Hamilton

Theorem Let A be an $n \times n$ matrix. Let $p = (-1)^n \lambda^n + c_1 \lambda^{n-1} + \cdots + c_n$ be its characteristic polynomial. Then

$$p(A) = (-1)^n A^n + c_1 A^{n-1} + \cdots + c_{n-1} A + c_n \Big(\cdots$$

is 0.

Linear maps version Let $f: V \to V$ be an endo of an n-dimensional vector space. Let

$$p_f = (-1)^n \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1} \lambda + c_n$$

be its characteristic polynomial. Then

$$p_f(f) = (-1)^n f^n + c_1 f^{n-1} + \cdots + c_{n-1} f + c_n \, id_V = 0.$$

Example $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

$$p = \lambda^2 - 2\lambda - 1$$

Check the theorem

$$A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix}, \quad \text{so}$$

$$\begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix} - 2\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$$

Before we prove the theorem recall

$$p = \det\left(A - \lambda \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}\right)$$

and that for any matrix we have

$$A \cdot \begin{pmatrix} \text{adjugate} \\ \text{matrix} \\ \tilde{A} \text{ of } A \end{pmatrix} = \det(A) \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

where

$$ij \text{ entry of } \tilde{A} = (-1)^{i+j} \det\begin{pmatrix} A \text{ but remove} \\ j\text{th row and} \\ i\text{th column} \end{pmatrix}$$

Let's apply this to $A - \lambda \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ to get

$$\left(A - \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}\right) \cdot \widetilde{\left(A - \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}\right)} \overset{(*)}{=} p(\lambda) \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Now $\widetilde{A - \lambda \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}$ is a matrix whose entries are polynomials of degree $\leq n-1$ in λ. Turning this around we get

$$\widetilde{A - \lambda \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}} = B_0 + \lambda B_1 + \dots + \lambda^{n-1} B_{n-1}$$

for some $n \times n$ matrices B_i. Going back to $(*)$ we see comparing coefficients of powers of λ:

$$-B_{n-1} = (-1)^n \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$-B_{n-2} + AB_{n-1} = c_1 \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$\vdots$$

$$-B_0 + AB_1 = c_{n-1} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$AB_0 = c_n \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

Solving, starting at the top we get

$$B_{n-1} = (-1)^{n-1}\begin{pmatrix}1 & & \\ & \ddots & \\ & & 1\end{pmatrix}$$

$$B_{n-2} = (-1)^{n-1}A - c_1\begin{pmatrix}1 & & \\ & \ddots & \\ & & 1\end{pmatrix}$$

$$\vdots$$

$$B_0 = (-1)^{n-1}A^{n-1} - c_1 A^{n-2} \cdots - c_{n-1}\begin{pmatrix}1 & & \\ & \ddots & \\ & & 1\end{pmatrix}$$

Then the last equation says

$$(-1)^{n-1}A^n - c_1 A^{n-1} \cdots - c_{n-1}A = c_n\begin{pmatrix}1 & & \\ & \ddots & \\ & & 1\end{pmatrix}$$

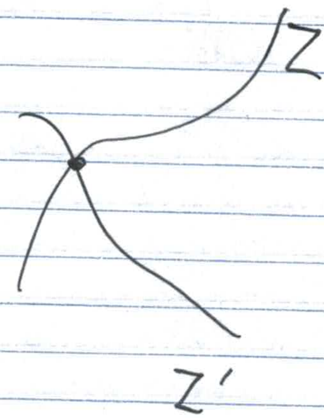
which is Cayley-Hamilton.

Z, Z' schemes $\quad i : Z \to X$, $i' : Z' \to X$ closed imm.

$$Z \amalg Z' \xrightarrow[\text{surj}]{} X \xrightarrow{\text{mono}} \mathrm{Spec}(A)$$

everything reduced.

$$\boxed{X = Z \coprod_{Z \cap Z'} Z'}$$



Picture for rings

$$\begin{array}{ccc} B \otimes_A B & & \\ \uparrow \uparrow & & \\ B & \twoheadrightarrow & C \times C' \\ \uparrow & & \\ A & & \end{array}$$

Pick $V \subset Z \cap Z'$ affine

Pick $U \subset Z$ ~~sh~~ affine

s.t. $W = U \cap (Z \cap Z') \subset V$

~~After shrinking~~

Pick $U' \subset Z'$ affine

s.t. $W' = U' \cap (Z \cap Z') \subset V$

Pick $h \in \Gamma(V, \mathcal{O}_V)$ s.t.

$$D(h) \subset W \cap W' \subset V$$

Pick $f \in \Gamma(U, \mathcal{O}_U)$ s.t.

$$f|_W = h|_W$$

Generalized Eigenspaces

Motivation Let $f: V \to V$ be an endo of an n-dimensional complex vector space. We can write

$$P_f = (-1)^n \prod_{i=1}^{r} (\lambda - \lambda_i)^{m_i}$$

where $m_1, \dots, m_r$ are algebraic multiplicites of the eigenvalues $\lambda_1, \dots, \lambda_r$.

By Cayley-Hamilton have

$$\prod_{i=1}^{r} (f - \lambda_i \, id_V)^{m_i} (v) = 0$$

for all $v \in V$. In particular, if there is only one eigenvalue λ_1 then $f - \lambda_1 id_V$ is nilpotent.

Definition

Let $f: V \to V$ as above and let $\lambda \in \mathbb{C}$ be an eigenvalue of f. Then we set

$$V_\lambda = \{ v \in V \mid \exists\, m \geq 0 \text{ s.t. } (f - \lambda)^m (v) = 0 \}$$

and we call it the <u>generalized eigenspace</u>

<u>Lemma 1</u>: $f(V_\lambda) \subset V_\lambda$

<u>Proof.</u> If $(f-\lambda id)^m(v)=0$ then

$$(f-\lambda id)^m f(v) = f(f-\lambda id)^m(v) = f(0)=0.$$

because f commutes with $f-\lambda id_V$ $\square$

<u>Lemma 2</u> If $\lambda \neq \mu$ then $V_\lambda \cap V_\mu = 0$.

<u>Proof.</u> Assume not true to get a contradiction. Say $v \in V_\lambda \cap V_\mu$ so $(f-\lambda id_V)^a(v)=0$ and $(f-\mu id_V)^b(v)=0$ for some $a, b \geq 1$. Then we can replace v by $(f-\lambda id_V)(v)$ if nonzero. Similarly we may ~~re~~replace v by $(f-\mu id_V)(v)$ if nonzero.

Hence may assume $a = b = 1$. Then

$f(v) = \lambda v = \mu v$ which is a

contradiction with $\lambda \neq \mu$ $\square$

<u>Lemma 3</u> $f - \lambda \, id_V : V_\mu \xrightarrow{\cong} V_\mu$ if $\lambda \neq \mu$.

<u>Pf</u> By Lemma 2 it is injective $\Rightarrow$ bijective

<u>Lemma 4</u>: If the eigenvalues of f are $\lambda_1, \dots, \lambda_r$ then

$$V = V_{\lambda_1} + \dots + V_{\lambda_r}$$

<u>Proof.</u> Say $v \in V$. Since

$$(f - \lambda_1)^{m_1} \cdots (f - \lambda_r)^{m_r}(v) = 0$$

by Cayley-Hamilton, we may

assume (by induction; details omitted) that

$$(f-\lambda_i)(v) \in V_{\lambda_1} + \cdots + V_{\lambda_r}$$

for some i. Say

$$(f-\lambda_i)(v) = v_1 + v_2 + \cdots + v_r$$

$$v_j \in V_{\lambda_j}.$$

For $j \neq i$ we can write $v_j = (f-\lambda_i)(v_j')$ for some $v_j' \in V_{\lambda_j}$ (Lemma 3). Then

$$(f-\lambda_i)(v - v_1' - \cdots - \widehat{v_i'} \cdots - v_r') \in V_{\lambda_i}.$$

$\overset{\text{def}^n}{\Longrightarrow}$ for some $m \geq 1$ we have
V_{λ_i}

$$(f-\lambda_i)^{m+1}(v - v_1' \cdots \widehat{v_i'} - v_r') = 0$$

$$\Longrightarrow \quad v - v_1' - \cdots \widehat{v_i'} - v_r' \in V_{\lambda_i}.$$

This is what we wanted to show. $\square$

Proof of Jordan normal form $\quad n = \dim V$

Say eigenvalues are $\lambda_1 \,--\, \lambda_r$

algebraic mult $\quad m_1 \,--\, m_r \quad \boxed{m_1 + \cdots + m_r = n}$

generalized eigenspaces $\quad V_{\lambda_1} \,--\, V_{\lambda_r} \quad \dim\ e_1, --, e_r$

$\left(\sum e_i \geq n \atop \text{Lemma 4}\right)$

We have seen $f - \lambda_1$ is nilpotent on V_{λ_1}

By last time we can pick a basis $v_1, \dots, v_{e_1}$ of V_{λ_1} such that matrix of $f - \lambda_1 |_{V_{\lambda_1}}$ looks like

$$
e_1 \updownarrow \begin{pmatrix} \boxed{J_1^0} & & & \\ & \boxed{J_2^0} & & \\ & & \ddots & \\ & & & \boxed{J_t^0} \end{pmatrix} \qquad J_k^0 = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}
$$

w.r.t. $v_1, -, v_{e_1}$. Then matrix of $f|_{V_{\lambda_1}}$

looks like

$$e_1 \updownarrow \begin{pmatrix} \boxed{J_1} & & \\ & \ddots & \\ & & \boxed{J_t} \end{pmatrix} \quad \text{with } J_k = \begin{pmatrix} \lambda_1 & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda_1 \end{pmatrix}$$

Extend to a $\overbrace{\text{basis of}}^{v_1 \dots v_n}$ V in some way.

We get: matrix of f looks like

$$e_1 \updownarrow \left(\begin{array}{ccc|c} \boxed{J_1} & & 0 & \\ & \ddots & & X \\ 0 & & \boxed{J_t} & \\ \hline & 0 & & *_{\text{left over}} \end{array}\right)$$

$\Longrightarrow$ char pol of f $= (\lambda_1 - \lambda)^{e_1} \cdot \begin{pmatrix} \text{char pol} \\ \text{of } *_{\text{Leftov}} \end{pmatrix}$

$\Longrightarrow e_1 \leq m_1$

$$\left.\begin{array}{l} n = m_1 + \cdots + m_r \\ n \leq e_1 + \cdots + e_r \\ e_i \leq m_i \end{array}\right\} \Longrightarrow e_i = m_i \quad \forall i$$

Now pick

$v_1^{(1)}, \ldots, v_{m_1}^{(1)}$ suitable basis (as above) for V_{λ_1}

$\vdots \qquad\qquad \vdots$

$v_1^{(r)}, \ldots, v_{m_r}^{(r)}$ suitable basis (as above) for V_{λ_r}

By equality of numbers we see

$v_1^{(1)}, \ldots, v_{m_1}^{(1)}, \ldots, v_1^{(r)}, \ldots, v_{m_r}^{(r)}$

basis of V and we're done

(OK this is a bit sketchy.) $\boxtimes$