EXERCISES

Contents

1. First problem set

Exercise 1.1. Let $a = (a_1, \dots, a_n) \in \mathbf{C}^n$. Recall that $ev_a : \mathbf{C}[x_1, \dots, x_n] \to \mathbf{C}$ is the map $f \mapsto f(a)$. Carefully prove that

(1) the ideal $(x_1 - a_1, \dots, x_n - a_n) \subset \mathbf{C}[x_1, \dots, x_n]$ is a maximal ideal, and
(2) we have $(x_1 - a_1, \dots, x_n - a_n) = \mathrm{Ker}(ev_a)$.

Exercise 1.2. Let R be any ring. Let $I \subset R$ be an ideal. Carefully prove that if $I \neq R$, then there exists a maximal ideal $\mathfrak{m} \subset R$ such that $I \subset \mathfrak{m}$.

Exercise 1.3. Consider the subset

$$X = \{(t, 1/t) \mid t \in \mathbf{C}, t \neq 0\} \subset \mathbf{C}^2$$

Show that this is an algebraic set by finding an ideal $I \subset \mathbf{C}[x, y]$ such that $X = V(I)$.

Exercise 1.4. Consider the subset

$$X = \{(4t^2 - 4t + 1, t^3 - 1) \mid t \in \mathbf{C}\} \subset \mathbf{C}^2$$

Show that this is an algebraic set by finding an ideal $I \subset \mathbf{C}[x, y]$ such that $X = V(I)$. (Hint: Write t in terms of x.)

Definition 1.5. Consider the affine space $\mathbf{C}^n$. An *affine line* in $\mathbf{C}^n$ is a translate of a linear subspace of dimension 1.

In other words, a line can be defined by a system of linear equations

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \cdots \\ a_{n-11}x_1 + \ldots + a_{n-1n}x_n = b_{n-1} \end{cases}$$

where the rank of the matrix (a_{ij}) is $n-1$. This makes it clear that an affine line is an algebraic subset. Of course we can also *parametrize* an affine line as

$$l = \{tv + w \mid t \in \mathbf{C}\}$$

where $v, w \in \mathbf{C}^n$, and $v \neq 0$.

Exercise 1.6. Let $X \subset \mathbf{C}^n$ be an algebraic subset. Let $l \subset \mathbf{C}^n$ be an affine line. Show that $X \cap l$ is either empty, or finite, or equal to l.

Exercise 1.7. Show that the subset

$$X = \{(t, e^t) \mid t \in \mathbf{C}\} \subset \mathbf{C}^2$$

is a usual closed set, but not an algebraic set.

Remark 1.8. More generally, if you have an algebraic set of the form $\{(t, f(t)) \mid t \in \mathbf{C}\}$ where $f : \mathbf{C} \to \mathbf{C}$ is a function what can say about the function f?

2. Second problem set

Exercise 2.1. Let $C \subset \mathbf{C}^2$ be a nonempty plane algebraic curve (remember this just means that C is a hypersurface in $\mathbf{C}^2$). Consider the map

$$C \longrightarrow \mathbf{C}, \quad (x, y) \longmapsto x$$

What can you say about the fibres of this map? More precisely, show the following:

(1) Show that if the fibre over $a \in \mathbf{C}$ is infinite, then $\{a\} \times \mathbf{C} \subset C$.
(2) Show that there exists an integer d such that all but finitely many fibres have cardinality d.
(3) Show that, with d as in (2), the other fibres have either $< d$ points, or are infinite.

Exercise 2.2. For which primes p do the polynomials $x^2 + 2x + 3 \bmod p$ and $4x^2 + 5x + 6 \bmod p$ have a root in common? (Hint: Compute the resultant of $x^2 + 2x + 3$ and $4x^2 + 5x + 6$. Please state the result you are using.)

The space of $n \times m$ (row $\times$ column) matrices is denoted $\mathrm{Mat}(n \times m, \mathbf{C})$. Of course we may think of this as copy of affine nm-space, because we can use the coefficients of the matrices to get a bijection

$$\mathrm{Mat}(n \times m, \mathbf{C}) \longrightarrow \mathbf{C}^{nm}, \quad A \longrightarrow (a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, \ldots, a_{nm})$$

In this way we can speak of algebraic sets in $\mathrm{Mat}(n \times m, \mathbf{C})$.

Exercise 2.3. Consider the subset

$$X = \{A \in \mathrm{Mat}(2 \times 2, \mathbf{C}) \mid A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\} \subset \mathrm{Mat}(2 \times 2, \mathbf{C}).$$

(1) Show that X is an algebraic set.
(2) Consider the map

$$\mathrm{Char} : \mathrm{Mat}(2 \times 2, \mathbf{C}) \longrightarrow \mathbf{C}^2, \quad A \longmapsto (\mathrm{Tr}(A), \det(A))$$

What is $\mathrm{Char}(X)$?

(3) What does your answer to (2) mean for the topology of X? (Just give the most obvious thing here.)

Since $\mathbf{C}^a \times \mathbf{C}^b = \mathbf{C}^{a+b}$ we can given $X \in \mathbf{C}^a$ and $Y \subset \mathbf{C}^b$ take their *product* $X \times Y$ in $\mathbf{C}^{a+b}$. It turns out that if X and Y are algebraic sets, then so is $X \times Y$. Namely, if $f_i \in \mathbf{C}[x_1, \ldots, x_n]$ define X and if $g_j \in \mathbf{C}[y_1, \ldots, y_m]$ define Y, then

$$X \times Y = V(f_i(x_1, \ldots, x_n), g_j(x_{n+1}, \ldots, x_{n+m}))$$

inside $\mathbf{C}^{a+b}$. Then X, Y and $X \times Y$ inherit the Zariski topology from $\mathbf{C}^a$, $\mathbf{C}^b$ and $\mathbf{C}^{a+b}$. Now, it is **not** true that that the Zariski topology on $X \times Y$ as defined just now is the product topology! This is clear on considering $X = \mathbf{C} \subset \mathbf{C}$, $X = \mathbf{C} \subset \mathbf{C}$ and $X \times Y = \mathbf{C}^2 \subset \mathbf{C}^2$ which clearly does not have the product topology (it has many more closed sets than just unions and intersections of products of closed subsets). What is true is that if X and Y are irreducible then $X \times Y$ is irreducible. Here are two exercises whose combination implies this fact.

Exercise 2.4. Let X, resp. Y be an algebraic set in $\mathbf{C}^a$, resp. $\mathbf{C}^b$.

(1) For every $x_0 \in X$ show that the map $Y \to X \times Y, y \mapsto (x_0, y)$ is continous in the Zariski topology.
(2) Show that the projection map
$$X \times Y \longrightarrow Y, \quad (x, y) \longmapsto y$$
is continuous in the Zariski topology.
(3) Combine (1) and (2) to show that the maps $Y \to X \times Y$, $y \mapsto (x_0, y)$, and $X \to X \times Y$, $x \mapsto (x, y_0)$ are homeomorphisms onto their image in the Zariski topology.
(4) Show that the projection map
$$X \times Y \longrightarrow Y, \quad (x, y) \longmapsto y$$
is open[1] in the Zariski topology. [Hint: If you have a continuous map of topological spaces $f : Z \to Y$ such that for every $z \in Z$ there exists a continuous map $\sigma : Y \to Z$ with $f \circ \sigma = \mathrm{id}_Y$, then f is open.]

In the following exercise we use the convention that an irreducible space is nonempty.

Exercise 2.5. Let $f : Z \to Y$ be a continuous map of topological spaces. Assume that

(1) f is open,
(2) Y is irreducible,
(3) for a dense set of points $y \in Y$ the fibre $f^{-1}(\{y\})$ is irreducible.

Show that Z is irreducible.

3. Third problem set

For an $n \times n$ matrix A over $\mathbf{C}$ we have the famous Cayley-Hamilton which says that

$$P(A) = 0$$

where $P(x) \in \mathbf{C}[x]$ is the characteristic polynomial of A, namely $P(x) = \det(x\mathbf{1}_{n\times n} - A)$. Here $\mathbf{1}_{n \times n}$ indicates the identity $n \times n$ matrix. Please use this in solving the exercise below.

[1]A continuous map of topological spaces is open if the image of an open set is open.

Exercise 3.1. Let R be a ring. Let $A = (a_{ij})$ be an $n \times n$ matrix with coefficients in R. Let $P(x) \in R[x]$ be the characteristic polynomial of A (defined as $\det(x\mathrm{id}_{n\times n} - A)$). Then $P(A) = 0$ in $\mathrm{Mat}(n \times n, R)$. [Hints: Prove it for the ring $\mathbf{Z}[a_{ij}]$ where you think of the a_{ij} as variables and as entries of the matrix A. Use that any polynomial ring $\mathbf{Z}[x_1, \ldots, x_N]$ is isomorphic to a subring of $\mathbf{C}$. Then use this to conclude for any R and A.]

The result of the preceding exercise says that if $P(x) = x^n + a_{n-1}x^n + a_{n-2}x^{n-2} + \ldots + a_0$ then the map

$$R^{\oplus n} \xrightarrow{A^n + a_{n-1}A^{n-1} + \ldots + a_0} R^{\oplus n}$$

is the zero map. Please use this in the exercise below.

Exercise 3.2. Suppose that $\varphi : R \to S$ is a ring map. Let $s_1, \ldots, s_n$ be elements of S which generate S as an R-module. Let $s \in S$ be an arbitrary element. By assumption for each i we can choose elements $a_{ij} \in R$ such that

$$ss_i = \sum\nolimits_{j=1,\ldots,n} \varphi(a_{ij})s_j$$

Denote A the $n \times n$ matrix over R whose coefficients are the elements a_{ij}. Let $P(x) \in R[x]$ be its characteristic polynomial. Show that on writing

$$P(x) = x^n + a_{n-1}x^n + a_{n-2}x^{n-2} + \ldots + a_0, \quad a_i \in R$$

we have

$$s^n + \varphi(a_{n-1})s^n + \varphi(a_{n-2})s^{n-2} + \ldots + \varphi(a_0) = 0$$

This means that s is integral over R, in other words this means that $R \to S$ is integral.

Exercise 3.3. Find the irreducible components of

$$V(xy, xz, yz) \subset \mathbf{C}^3$$

where we use x, y, z as coordinates on $\mathbf{C}^3$ instead of the usual x_1, x_2, x_3.

Exercise 3.4. Find the irreducible components of

$$V(xy - z^2, x^2 + y^2 + z^2) \subset \mathbf{C}^3$$

where we use x, y, z as coordinates on $\mathbf{C}^3$ instead of the usual x_1, x_2, x_3.

Exercise 3.5. [2] Find the irreducible components of

$$V(x^2 + y^2 + z^2 + 1, x^2 + y^2 + z^2 + 2) \subset \mathbf{C}^3$$

where we use x, y, z as coordinates on $\mathbf{C}^3$ instead of the usual x_1, x_2, x_3.

Exercise 3.6. [3] Find the irreducible components of

$$V(x^2 + 2y^2 + 3z^2 + 2, x^2 + 3y^2 + 2z^2 + 1) \subset \mathbf{C}^3$$

where we use x, y, z as coordinates on $\mathbf{C}^3$ instead of the usual x_1, x_2, x_3. (Hint: This one is pretty hard. Try to eliminate a variable after changing coordinates.)

[2] In the actual homework there was a typo, namely the second equation had a z^3 instead of z^2.
[3] In the actual homework there was a typo, namely the second equation had a z^3 instead of z^2.

4. Fourth problem set

Some questions about dimension.

Exercise 4.1. Let $X \subset Y \subset \mathbf{C}^n$ be affine algebraic subsets. Show that $\dim(X) \leq \dim(Y)$.

Exercise 4.2. Let $X \subset \mathbf{C}^n$ be a linear subspace. Show that X is a variety and that the dimension of X as a variety is the same as the dimension of X as a linear space.

Exercise 4.3. Let $X \subset \mathbf{C}^n$ and $Y \subset \mathbf{C}^m$ be affine varieties. Let $f_j(x_1, \ldots, x_n)$, $j = 1, \ldots, m$ be polynomials. Consider the map

$$f : \mathbf{C}^n \longrightarrow \mathbf{C}^m, \quad (a_1, \ldots, a_n) \longmapsto (f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n))$$

and assume

(1) $f(X) \subset Y$, and
(2) $f(X)$ is Zariski dense in Y.

Prove that $\dim(X) \geq \dim(Y)$. (Hint: Consider the associated map of coordinate rings $\Gamma(Y) \to \Gamma(X)$, see Fulton.)

Exercise 4.4. Let $C = V(f) \subset \mathbf{C}^2$ be an irreducible affine plane curve. Consider the projection

$$\pi : C \longrightarrow \mathbf{C}, \quad (x, y) \longrightarrow x$$

Prove the following are equivalent:

(1) the corresponding ring map $\mathbf{C}[x] \to \Gamma(C)$ is finite,
(2) the polynomial f can be written as $\lambda y^d + \sum_{i<d} a_i(x) y^i$ with $\lambda \in \mathbf{C}$ not zero,
(3) the map $\pi : C \to \mathbf{C}$ of usual topological spaces is proper.

Hints: In the course we proved that (1) implies (3). Prove (1) is equivalent to (2) by doing some algebra. To prove that (3) implies (2) try to show that if $f = a_d(x) y^d + \sum_{i<d} a_i(x) y^i$ and $z \in \mathbf{C}$ is a zero of $a_d(x)$, then the solutions of $f(z', y) = 0$ are unbounded as $z' \to z$.

Definition 4.5. Let $C \subset \mathbf{C}^2$ be an irreducible plane curve. We say a (linear) projection $\pi : C \to \mathbf{C}$, i.e., a map coming from a nonzero linear map $\mathbf{C}^2 \to \mathbf{C}$, is *finite* if the equivalent conditions of Exercise 4.4 hold (after suitably changing coordinates for conditions (1) and (2)).

All the linear projections $\mathbf{C}^2 \to \mathbf{C}$ can be parametrized (up to linear coordinate changes) by an element $s \in \mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$. Namely, to a slope $s \in \mathbf{C}$ we associate the projection

$$p_s : \mathbf{C}^2 \longrightarrow \mathbf{C}, \quad (x, y) \longmapsto sx - y$$

and to the slope $s = \infty$ we associate the projection

$$p_\infty : \mathbf{C}^2 \longrightarrow \mathbf{C}, \quad (x, y) \longmapsto x.$$

Exercise 4.6. Let $C = V(1 + x^3 + y^3) \subset \mathbf{C}^2$. This is an irreducible affine plane curve (you can use this). Which of the projections p_s given above define a finite morphism

$$\pi = p_s|_C : C \longrightarrow \mathbf{C}?$$

Explain your answer. More generally, if $C = V(f)$ which projections p_s give a finite morphism $C \to \mathbf{C}$? Only give an answer, no need to explain.

5. Fith problem set

Let R be a ring. Let $f \in R$ be an element. Any of the following symbols

$$R[1/f] = R_f = \{1, f, f^2, f^3, \ldots\}^{-1}R = R[x]/(fx - 1)$$

will denote the ring constructed out of R and f in the following manner. An element of $R[1/f]$ is a *fraction* a/f^n with $n \geq 0$ and $a \in R$. We identify two fractions a/f^n and b/f^m if we can find an integer N such that $f^N(f^m a - f^n b) = 0$ in R. You can check that this defines an equivalence relation. If R is a domain (and $f \neq 0$) then this is equivalent to asking $f^m a = f^n b$, which is the thing you are used to. We add and multiply fractions by the rules

$$a/f^n + b/f^m = (af^m + bf^n)/f^{nm}, \quad a/f^n \cdot b/f^m = ab/f^{nm}.$$

Then it is easy to verify that this is a ring with zero $0/1$ and unit $1/1$. The map $R \to R_f$, $a \mapsto a/1$ is a ring map with the pleasing property that f maps to an invertible element. In fact the ring map $R \to R_f$ is universal among all ring maps $R \to A$ which map f to an invertible element. The identification of R_f with the ring $R[x]/(fx - 1)$ uses the maps

$$R_f \to R[x]/(xf - 1),\ a/f^n \mapsto ax^n, \quad R[x]/(xf - 1) \to R_f,\ ax^n \mapsto a/f^n.$$

Finally, if R is a domain, then the ring R_f is simply the subring of the quotient field of R consisting of all fractions whose denominator is a power of f.

Exercise 5.1. Let $f \in \mathbf{C}[x_1, \ldots, x_n]$ be nonzero. Let

$$X = \mathbf{C}^n \setminus V(f) = \{a \in \mathbf{C}^n \mid f(a) \neq 0\}$$

Show that the ring of regular functions on X can be described as follows

$$\mathcal{O}(X) = \mathbf{C}[x_1, \ldots, x_n][1/f] = \mathbf{C}[x_1, \ldots, x_n]_f.$$

(This may have been explained in the lectures, but please rewrite it anyway.)

Exercise 5.2. Let $f, g \in \mathbf{C}[x_1, \ldots, x_n]$ be nonzero. Assume that $V(f)$ and $V(g)$ have no irreducible component in common. Set

$$X = \mathbf{C}^n \setminus V(f, g).$$

Show that

$$\mathcal{O}(X) = \mathbf{C}[x_1, \ldots, x_n].$$

Hint: Use previous exercise.

Exercise 5.3. Consider the affine plane curve

$$C = \{(x, y) \mid y^2 = x(x - 1)(x + 3)\}$$

and the point $c = (-1, 2) \in C$. Let $U = C \setminus \{c\}$ which is a quasi-affine variety. Find an element of $\mathcal{O}(U)$ which is not an element of $\mathcal{O}(C) = \Gamma(C)$.

Exercise 5.4. Consider the cuspidal curve $C_{cusp} = \{(x, y) \in \mathbf{C}^2 \mid y^2 - x^3 = 0\} \subset \mathbf{C}$ and the affine line $C_{smooth} = \{t \in \mathbf{C} \mid 1 = 1\} = \mathbf{C}$. Consider the map

$$\varphi : C_{smooth} \longrightarrow C_{cusp},\ t \mapsto (t^2, t^3)$$

Note that φ is bijective. Show that its inverse $\varphi^{-1} : C_{cusp} \to C_{smooth} = \mathbf{C}$ is not a regular function.

Exercise 5.5. Consider plane conics

$$C = \{(x, y) \in \mathbf{C}^2 \mid a + bx + cy + dx^2 + exy + fy^2 = 0\}.$$

over the complex numbers where we assume at least one of d, e, f is nonzero. For which $(a, b, c, d, e, f) \in \mathbf{C}^6$ does there exist a "parametrization" $\varphi : \mathbf{C} \to C$? Here "parametrization" means

(1) $\varphi(t) = (P(t), Q(t))$ for some polynomials P, Q,
(2) φ is bijective, and
(3) the inverse map φ^{-1} is a regular function on C.

Hint: Remember your classification of conics...!

6. Sixth problem set

Exercise 6.1. Let $X \subset \mathbf{C}^n$ be an affine variety. Let $f \in \mathbf{C}[x_1, \ldots, x_n]$ be a polynomial such that $X \not\subset V(f)$.

(1) Show that
$$X \setminus V(f) \longrightarrow \mathbf{C}^{n+1}, \quad a \longmapsto (a_1, \ldots, a_n, 1/f(a))$$
is a morphism.
(2) Show that the image is a Zariski closed $Y \subset \mathbf{C}^{n+1}$.
(3) Show that the induced map $X \setminus V(f) \to Y$ is bijective.
(4) Show that the inverse map $Y \to X \setminus V(f)$ is a morphism too.

Conclude that $X \setminus V(f)$ is an affine variety (as redefined in the course).

Exercise 6.2. Let $X \subset \mathbf{C}^n$ be a quasi-affine variety (as defined in the course). Let $a \in X$ be a point. Show that there exists an open neighbhourhood $U \subset X$ of a which is an affine variety (as redefined in the course). (Hint: Use previous exercise.)

Let $X \subset \mathbf{C}^n$ be an affine variety. Let $a = (a_1, \ldots, a_n) \in X$ be a point. Let $I = I(X)$ be the ideal of X. Then we have $f(a) = 0$ for all $f \in I$. But it is usually not the case that

$$\frac{\partial f}{\partial x_i}(a) = 0$$

for $f \in I$. Hence we get an interesting $\mathbf{C}$-linear map

$$I \longrightarrow \mathbf{C}^n, \quad f \longmapsto \left(\frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a)\right) \tag{6.2.1}$$

We say a is a *nonsingular* point of X if and only if $\mathrm{rank}(6.2.1) = n - \dim(X)$. We say a is a *singular* point of X if and only if $\mathrm{rank}(6.2.1) < n - \dim(X)$. It is a theorem in commutative algebra that the rank is never $> n - \dim(X)$, so that this covers all cases.

Exercise 6.3. Let $X = V(x_1^2) \subset \mathbf{C}^n$. What are the singular points of X? (This is a trick question. Think, don't compute.)

Exercise 6.4. Let

$$X = V(x_1^2 + x_2^2 + \ldots + x_n^2) \subset \mathbf{C}^n.$$

What are the singular points of X? (Compute, don't think.)

Exercise 6.5. Let $f \in \mathbf{C}[x, y]$ be irreducible, so that $C = V(f) \subset \mathbf{C}^2$ is a plane curve What equations define the singular points of C? Can there be infinitely many singular points?

Exercise 6.6. Consider the affine curve $C \subset \mathbf{C}^3$ defined by the ideal

$$I = ((x-1)^2 + y^2 + z^2 - 1, (x+1)^2 + 2 * y^2 + z^2 - 1) \subset \mathbf{C}[x, y, z]$$

Compute its singular points. (You may use that I is a prime ideal and that $C = V(I)$ is indeed a curve.)

7. Seventh problem set

Please, please use any of the statements from the list in Section 100. If you need an extra general statement, then email it to me and I'll add it to the list.

Exercise 7.1. Compute $\mathcal{O}(\mathbf{P}^1)$.

Exercise 7.2. Is there a nonconstant morphism $\mathbf{P}^1 \to \mathbf{C}^n$ for any n?

Exercise 7.3. Compute $\mathcal{O}(\mathbf{P}^2 \setminus V_+(X_0 + X_1 + X_2))$.

Exercise 7.4. Compute $\mathcal{O}(\mathbf{P}^2 \setminus V_+(X_0^2 + X_1^2 + X_2^2))$.

Exercise 7.5. Let $F_0, F_1, F_2 \in \mathbf{C}[X_0, X_1]$ be homogeneous of the same degree. Assume that

$$V_+(F_0) \cap V_+(F_1) \cap V_+(F_2) = \emptyset.$$

Show that the map

$$\mathbf{P}^1 \longrightarrow \mathbf{P}^2, \quad [a_0 : a_1] \mapsto [F_0(a_0, a_1) : F_1(a_0, a_1) : F_2(a_0, a_1)]$$

is a morphism of varieties. Generalize the statement to higher dimensional projective spaces (but don't prove it).

Exercise 7.6. Find two (nondegenerate) conics in $\mathbf{P}^2$ which meet in one point. Find two irreducible cubics in $\mathbf{P}^2$ which meet in one point.

8. Eighth problem set

Review of holomorphic functions. Let $\Omega \subset C$ be an open subset. A function $f : \Omega \longrightarrow \mathbf{C}$ is called *holomorphic* if for every $z_0 \in \Omega$ there exists a complex number $f'(z_0)$, called the *derivative of f at z_0* such that

$$\forall \epsilon > 0 \ \exists \delta > 0 : \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon \ \forall z \in \Omega, \ |z - z_0| < \delta.$$

These functions have the following properties:

(1) Given complex numbers $a_0, a_1, a_2, \ldots$ such that $|a_n| < C(1/R)^n$ for some $C, R > 0$ then the powerseries $f(z) = \sum a_n(z - z_0)^n$ converges for all z, $|z - z_0| < R$ and is a holomorphic function on that disc.
(2) Conversely, if $f : \Omega \to \mathbf{C}$ is holomorphic, and a disc of radius R around z_0 is contained in Ω, then f is given by a convergent power series on the disc with radius R around z_0 as in (1).
(3) If f is holomorphic, then f is continuous.
(4) If f is holomorphic, then the derivative $z \mapsto f'(z)$ is holomorphic too.
(5) The derivative of $f(z) = \sum a_n(z - z_0)^n$ as in (1) is $f'(z) = \sum n a_n (z - z_0)^{n-1}$ which converges on the same disc.

(6) If $f : \Omega \to \mathbf{C}$ is holomorphic and nonconstant on each connected component of Ω, then f is an open mapping.
(7) If f, g are holomorphic on Ω and $f(z_n) = g(z_n)$ for some infinite set of points $\{z_n\} \subset \Omega$ which has a limit point in Ω then $f = g$.
(8) Let f_i, $i = 1, 2, 3, \ldots$ be holomorphic functions defined on the open subset $\Omega \subset \mathbf{C}$. Assume the pointwise limit $f(z) = \lim_i f_i(z)$ exists for all $z \in \Omega$. Assume moreover that the convergence $\lim f_i = f$ is *uniform* on every compact subset of Ω. Then f is holomorphic too.

Theorem 8.1. *Statement of the implicit function theorem as we proved it in the lectures*[4]. *Suppose that $C \subset \mathbf{C}^n$ is a quasi-affine curve. Let $p = (a_1, \ldots, a_n) \in C$ be a nonsingular point of C. Then there exist*

(1) *an $i \in \{1, \ldots, n\}$,*
(2) *$\epsilon > \delta > 0$, and*
(3) *functions $g_1, \ldots, g_n : \{z \in \mathbf{C} : |z| < \delta\} \to \mathbf{C}$*

such that the following conditions hold

(1) *$g_i(z) = z + a_i$,*
(2) *$g_1, \ldots, g_n$ are holomorphic,*
(3) *$|g_j(z) - a_j| < \epsilon$ for all j and all $|z| < \delta$,*
(4) *we have*

$$C \cap \{(z_1, \ldots, z_n) : |z_j - a_j| < \epsilon, |z_i - a_i| < \delta\} = \{(g_1(z), \ldots, g_n(z)) : |z| < \delta\}$$

What this means geometrically is the following: Consider the projection $\pi_i : X \to \mathbf{C}$ to the ith axis. For a sufficiently small open neighbourhood $U \subset \mathbf{C}^n$ (above this is the ball of radius ϵ) of p, there exists a small open neighbourhood $\Omega \subset \mathbf{C}$ (this is the disc of radius δ) of a_i such that

$$\pi_i : X \cap U \cap \pi_i^{-1}(\Omega) \longrightarrow \Omega$$

is bijective, with inverse $\Phi : z \mapsto (g_1(z), \ldots, g_n(z))$ whose components are holomorphic. A reformulation which is easier to parse is the following. (Here we reparametrize the disc $|z| < \delta$ to convert it to the unit disc.)

[4] I owe you the proof of the uniqueness of the solution. Suppose that $\varphi_1, \ldots, \varphi_{n-1} \in \mathbf{C}[x_1, \ldots, x_n]$ are as in the lecture, i.e., have no linear or constant terms. Pick a constant $C_1 > 0$ such that

$$|\varphi_j(x_1 + y_1, \ldots, x_n + y_n) - \varphi(x_1, \ldots, x_n)| < C(\max\{|x_i|\} + \max\{|y_j|\}) \max\{|y_i|\}$$

for all $x_i, y_i \in \mathbf{C}$ with $|x_i|, |y_i| \leq 1$. This constant is slightly different from the constant in Lemma 1 of the lecture. Let $|z| < \epsilon$. Suppose that we have $y_1, \ldots, y_{n-1}, y'_1, \ldots, y'_{n-1} \in \mathbf{C}$ with $|y_j|, |y'_j| < \epsilon$ and $y_j = \varphi_j(y_1, \ldots, y_{n-1}, z)$, and $y'_j = \varphi_j(y'_1, \ldots, y'_{n-1}, z)$. Set $\delta_j = y'_j - y_j$. Then

$$\begin{aligned} |\delta_j| = |y'_j - y_j| &= |\varphi_j(y'_1, \ldots, y'_{n-1}, z) - \varphi_j(y_1, \ldots, y_{n-1}, z)| \\ &= |\varphi_j(y_1 + \delta_1, \ldots, y_{n-1} + \delta_{n-1}, z) - \varphi_j(y_1, \ldots, y_{n-1}, z)| \\ &\leq C_1(\max\{|y_j|, |z|\} + \max\{|\delta_j|\}) \max\{|\delta_j|\} \end{aligned}$$

with C_1 as above. Now note that this is a contradiction as soon as ϵ is small enough (Hint: $|\delta_j| \leq 2\epsilon$; for example $4\epsilon C_1 < 1$ is good enough). In this arguement I did not have to impose a stronger condition on $|z|$ to make this work. But in the statement of the theorem I do need to choose $0 < \delta < \epsilon$ because in the theorem we are working with a general coordinate system, and then the trick with requiring $|z| < \delta$ is necessary, see Exercise 8.3. The discrepancy happens in the translation of the general result into the special coordinate system.

Theorem 8.2. *Let $p \in C \subset \mathbf{C}^n$ a nonsingular point of an algebraic curve. There exists a map*

$$\Phi : \{z \in \mathbf{C} : |z| < 1\} \longrightarrow C, \quad z \longmapsto (g_1(z), \dots, g_n(z))$$

such that: (a) each g_i is holomorphic, (b) $\Phi(0) = p$, (c) $g_i'(0) \neq 0$ for some i, and (d) Φ induces a homeomorphism of the disc $\{z \in \mathbf{C} : |z| < 1\}$ with a usual open neighbourhood of p in C.

Exercise 8.3. Show that the statement of Theorem 8.1 is false if you try to take $\epsilon = \delta$ in it. Namely consider the curve

$$C = \{(t + t^2, t - t^2), t \in \mathbf{C}\} = \{(x, y) \mid 2y - 2x + x^2 + 2xy + y^2 = 0\}$$

Show that there does not exist an $\epsilon < 1$ such that

$$\{(x, y) \in C : |x| < \epsilon\} = \{(x, y) \in C : |x| < \epsilon \text{ and } |y| < \epsilon\}$$

and similarly that there does not exist an ϵ such that

$$\{(x, y) \in C : |y| < \epsilon\} = \{(x, y) \in C : |x| < \epsilon \text{ and } |y| < \epsilon\}$$

Hint: Look at real t!

Exercise 8.4. Let $f \in \mathbf{C}[x, y]$. Assume that $f = xy + h.o.t.$, in other words f has no constant and linear terms and its quadratic term is xy. (For example $f = xy + x^3 + y^3$). Show that there exist holomorphic functions $g_1, g_2 : \{z \in \mathbf{C} : |z| < 1\}$ such that

1. $f(g_1(z), g_2(z)) = 0$ for all $|z| < 1$,
2. $g_1'(0) \neq 0$.

Hint: Apply Theorem 8.2 to the algebraic curve given by $x^{-2} f(x, xy) = 0$ and the point $p = (0, 0)$.

Consider a sphere of radius ϵ in $\mathbf{C}^2$. It is given by

$S^3_\epsilon = \{(z_1, z_2) \in \mathbf{C}^2 : z_1\overline{z_1} + z_2\overline{z_2} = \epsilon\} = \{(x_1, y_1, x_2, y_2) \in \mathbf{R}^4 : x_1^2 + y_1^2 + x_2^2 + y_2^2 = \epsilon\}$

and we have a stereographic projection

$$S^3_\epsilon \setminus \{northpole\} \longrightarrow \mathbf{R}^3, \quad (x_1, y_1, x_2, y_2) \longmapsto \left(\frac{x_1}{1 - y_2}, \frac{y_1}{1 - y_2}, \frac{x_2}{1 - y_2}\right)$$

Hence if we have a subset of the sphere then we may think of it as a subset of $\mathbf{R}^3 \cup \{\infty\}$.

Exercise 8.5. Consider an algebraic curve $C \subset \mathbf{C}^2$. Assume that $(0, 0) \in C$ is a smooth point. Convince yourself that for any small enough ϵ

$$C \cap S^3_\epsilon$$

is (topologically) a single unknotted loop. Things you should do here: (a) do a straightforward example, (b) do a less straightforward example, and (c) give an idea of how you could use Theorem 8.2 to start proving this (for example translate it into a question about holomorphic functions which seems reasonable to you).

Exercise 8.6. Consider the algebraic set $X = V(xy) \subset \mathbf{C}^2$. Show (by a calculation and picture) that for any small enough ϵ

$$X \cap S^3_\epsilon$$

is (topologically) two circles and show that they are linked (I mean that if they were made out of steel then you couldn't separate them).

Exercise 8.7. Consider the algebraic curve $C = V(y^2 - x^3) \subset \mathbf{C}^3$. Show (by a calculation and picture) that for any small enough ϵ

$$C \cap S^3_\epsilon$$

is (topologically) a single loop which is knotted.

Exercise 8.8. Let $f = \sum_{i \geq 0} a_i z^i$ be a power series with complex coefficients. Suppose that $a_0 \neq 0$. Let $n \geq 1$. Show that there exists a power series $g = \sum_{i \geq 0} b_i z^i$ such that $f = g^n$.

9. Ninth problem set

Suppose that $X \subset \mathbf{C}^n$ is an affine algebraic variety. Let $\tilde{f} \in \mathbf{C}[x_1, \ldots, x_n]$ be an element such that $X \not\subset V(\tilde{f})$. Then in Exercise 6.1 we have seen that $U = X \setminus V(\tilde{f})$ is an affine variety. Let $f \in \mathcal{O}(X)$ denote the restriction of $\tilde{f}$ to X. Then f restricts to an invertible element of $\mathcal{O}(U)$. Hence the restriction induces a canonical map

$$\mathcal{O}(X)_f \longrightarrow \mathcal{O}(U).$$

This map is an isomorphism. Actually, in the course of doing Exercise 6.1 you most likely proved this along the way, but it wasn't clearly stated as such. Also, in Exercise 5.1 you proved this when $X = \mathbf{C}^n$.

Exercise 9.1. Let X be an affine algebraic variety. Let $V \subset X$ be a nonempty Zariski open which is an affine variety also. Show that the restriction mapping $\mathcal{O}(X) \to \mathcal{O}(V)$ induces an isomorphism of fraction fields. Hint: Use remarks above and a suitable choice of an open $U \subset V \subset X$.

Exercise 9.2. In the lectures we classified all discrete valuations on $\mathbf{C}(x)/\mathbf{C}$ by a direct argument.

(1) Let $K = \mathbf{C}(x)[y]/(y^2 - x(x-1)(x-2))$. Classify all discrete valuations on $K/\mathbf{C}$ using a direct argument.
(2) Suppose $K = \mathbf{C}(x)[y]/(y^2 - x(x-1)(x-2)(x-3))$. What happens with the discrete valuations "at ∞" in this case?

Exercise 9.3. Give an example of a finite extension $\mathbf{C}(x) \subset K$ which is not cyclic; for example an extension which is not Galois or an extension which is Galois but whose Galois group is not cyclic. (This exercise is here to convince you that $\mathbf{C}(x)$ is very different from $\mathbf{C}((x))$ which has only cyclic finite extensions.)

Exercise 9.4. Let $P = [0 : 0 : 1] \in \mathbf{P}^2$. Consider projection from P which is the morphism of quasi-projective varieties

$$\pi : \mathbf{P}^2 \setminus \{P\} \longrightarrow \mathbf{P}^1, \quad [a_0 : a_1 : a_2] \longmapsto [a_0 : a_1].$$

Show that every fibre of π is isomorphic (as a variety) to the affine curve $\mathbf{C}$.

Exercise 9.5. Let $C = V_+(F) \subset \mathbf{P}^2$ be projective plane curve of degree d. This means that $F \in \mathbf{C}[X_0, X_1, X_2]$ is irreducible and homogeneous of degree d. Assume that $P = [0 : 0 : 1] \notin C$. Consider the restriction of the projection π of Exercise 9.4 to C, which is a morphism $\pi|_C : C \to \mathbf{P}^1$. Show that

(1) $\pi|_C : C \to \mathbf{P}^1$ is a proper map on underlying usual topological spaces, and
(2) for all but finitely points in $\mathbf{P}^1$ the fibre of $\pi|_C$ has exactly d points.

Hints: For (1) use results from Section 4. For (2) use the result of Exercise 2.1.

10. Tenth problem set

Let $A \to B$ be a ring map. The *integral closure* B' of A in B is the subset

$$B' = \{x \in B \mid x \text{ is integral over } A\}.$$

Exercise 10.1. Show that B' is an A-subalgebra of B. Hint: The difficult step is to show $x, y \in B' \Rightarrow x + y, xy \in B'$. To see this show that the A-subalgebra B'' of B generated by x and y is finite over A, and apply the result of Exercise 3.2.

Exercise 10.2. Let $K \subset L$ be a finite separable field extension generated by a single element. So $L = K[y]/(P(y))$ where $P \in K[T]$ is a monic irreducible polynomial with $\gcd(P, P') = 1$. Here $P' = \mathrm{d}P/\mathrm{d}T$. Note that this implies $P'(y)$ is not zero in L. Show that

$$\mathrm{Tr}_{L/K}\left(\frac{y^i}{P'(y)}\right) = \begin{cases} 0 & \text{if} \quad i = 0, \ldots, n-2 \\ 1 & \text{if} \quad i = n-1 \end{cases}$$

where $n = \deg_T(P) = [L : K]$. Use the following steps (or if you have a different proof that would be great too):

(1) Let $\overline{K}$ be an algebraic closure of K and write $P(T) = (T - \alpha_1)(T - \alpha_2)\ldots(T - \alpha_n)$ with $\alpha_1, \ldots, \alpha_n \in \overline{K}$. The fact that P is a separable polynomial means that $\alpha_i \neq \alpha_j$ for $i \neq j$.

(2) Let $\beta \in L$ be any element. Represent β as the congruence class of $Q(y)$ for some polynomial $Q(T) \in K[T]$. Show that

$$\mathrm{Tr}_{L/K}(\beta) = \sum\nolimits_{j=1,\ldots,n} Q(\alpha_j).$$

It is OK to find this in a book and refer to it.

(3) Show that

$$\mathrm{Tr}_{L/K}\left(\frac{y^i}{P'(y)}\right) = \sum\nolimits_{j=1,\ldots,n} \frac{\alpha_j^i}{P'(\alpha_j)}$$

(4) Show that

$$P'(\alpha_j) = (\alpha_j - \alpha_1)\ldots(\widehat{\alpha_j - \alpha_j})\ldots(\alpha_j - \alpha_n)$$

(5) Show that

$$\frac{1}{P(T)} = \sum\nolimits_{j=1,\ldots,n} \frac{1}{P'(\alpha_j)(T - \alpha_j)}$$

(6) Take the previous expression and do Taylor expansion in $1/T$ to conclude.

In the following exercises you may use the following fact that was proved in the lecture by Jarod Alper: Suppose that we have

$$A = \mathbf{C}[x] \subset B = \mathbf{C}[x, y]/(P)$$

where P is a polynomial in x, y which is monic as a polynomial in y and irreducible. Let B' be the integral closure of A in the fraction field of the domain B. Then we have

$$B \subset B' \subset \frac{1}{P'}B$$

where $P' = \partial P/\partial y$.

Exercise 10.3. Let $f \in \mathbf{C}[x]$ be a nonconstant polynomial which is not a square. This implies that $P = y^2 - f$ is irreducible. Let $A = \mathbf{C}[x]$ and $B = \mathbf{C}[x, y]/(P)$ as above. Show the following:

(1) The integral closure of A in the fraction field of B is $\mathbf{C}[x,z]/(z^2-g)$, where g is the square free part[5] of f.
(2) The ring B is integrally closed its fraction field if and only if f is square free.

Exercise 10.4. Let $f = (x-1)x^2(x+1)^3$. Then $P = y^3 - f$ is irreducible. Let $A = \mathbf{C}[x]$ and $B = \mathbf{C}[x,y]/(P)$ as above. Compute the integral closure of A in the fraction field of B.

11. Eleventh problem set

Catch up with homeworks you are behind on.

12. Twelth problem set

Exercise 12.1. (Krull intersection theorem.) Let R be a Noetherian domain. Prove the following statements (please skip the ones that you think are too easy):

(1) If I, J are ideals in R and for every $x \in I$ there exists an $n > 0$ such that $x^n \in J$, then $I^N \subset J$ for some $N \geq 1$.
(2) Let I' be an ideal of R and $x \in R$. Set $I'_n = \{y \in R \mid yx^n \in I'\}$. Show there exists a k such that $I'_k = I'_m$ for all $m \geq k$.
(3) Let I, J be ideals of R. Consider the set of ideals I' of R such that $I' \cap J \subset IJ$. Show that this set ordered by inclusion has a maximal element. (Hint: Zorn's lemma.)
(4) If I, J are ideals of R then $I^n \cap J \subset IJ$ for some n. [Hints: Let I' be maximal with $I' \cap J \subset IJ$ as in part (3). Show that $(I' + IJ) \cap J \subset IJ$ too, so $I' = I' + IJ$ by maximality. Hence $I' \cap J = IJ$. By part (1) it suffices to show that any $x \in I$ has a power which lies in I'. Let pick k exactly as in (2). Consider $I'' = I' + x^kR$. Check that $I'' \cap J \subset IJ$ by a clever little argument. Hence $I' = I''$, hence $x^k \in I'$ as desired.]
(5) Conclude that $\bigcap_{n \geq 0} I^n = 0$ if $I \neq R$. [Hint: If $x \in \bigcap I^n$, then use (4) to show that $x \in I^n \subset xI$ for some n which gives $x(1-f) = 0$ for some $f \in I$.]

This argument is from a paper by Karamzadeh. But I'm sure there are lots of other even more elementary arguments. Actually I just found one. It is an argument of H. Perdy and you can find it in his paper "An Elementary Proof of Krull's Intersection Theorem" published in the The American Mathematical Monthly, Vol. 111, No. 4 (Apr., 2004), pp. 356-357. If you want to look up his argument and explain it then that is fine too (and it will probably save you quite a bit of time). Note: it doesn't prove part (4) which is interesting in itself. Part (4) is a special case of the Artin-Rees theorem.

The rest of the exercises is a series of exercises aimed in some sense at understanding the "points at infinity". While doing them you will also be reviewing some of the material we've treated in the lectures. This means that some of the questions are formulated in a somewhat strange manner.

[5] For example the square free part of $f = (x-1)^3(x-2)^2$ is $g = (x-1)$.

Exercise 12.2. Consider a situation

$$\begin{tikzcd} & \hat{B} \arrow[r] & \hat{L} & \\ \mathbf{C}[[x]] \arrow[r, equal] & \hat{A} \arrow[u] \arrow[r] & \hat{K} \arrow[u] \arrow[r, equal] & \mathbf{C}((x)) \end{tikzcd}$$

where $\hat{K} \subset \hat{L}$ is a finite ring extension, the ring $\hat{L}$ is reduced, and $\hat{B}$ is the integral closure of $\hat{A}$ in $\hat{L}$. Recall that this means we can find compatible isomorphisms

$$\begin{tikzcd} \hat{B} \arrow[r] & \hat{L} \\ \prod_{i=1,\ldots,r} \mathbf{C}[[y_i]] \arrow[u, "\cong"] \arrow[r] & \prod_{i=1,\ldots,r} \mathbf{C}((y_i)) \arrow[u, "\cong"'] \end{tikzcd}$$

with moreover $x \mapsto (y_1^{e_1}, \ldots, y_r^{e_r})$. Prove the following: Given $r+1$ units

$$u_1, \ldots, u_{r+1} \in \hat{L}^*$$

of $\hat{L}^*$ there exist integers $m_1, \ldots, m_{r+1} \in \mathbf{Z}$ not all zero such that $u = u_1^{m_1} \ldots u_{r+1}^{m_{r+1}}$ is an element of $\hat{B}^*$. (In other words, u no longer has a "pole" at any of the "points" lying over $x = 0$.)

Exercise 12.3. Let C be a normal affine algebraic curve. Recal that this means that the ring of regular functions $B = \mathcal{O}(C)$ is a normal domain. More precisely, if $L = \mathbf{C}(C) = f.f.(B)$ is the field of rational functions of C, then B is integrally closed in L. We have also seen that C is a nonsingular curve. Moreover, we proved that there exists a ring map $\mathbf{C}[x] = A \to B$ such that B is finite over A (Noether Normalization – works even for nonnormal affine curves). Fix such a choice. Denote $K = \mathbf{C}(x) \subset L$. Diagram

$$\begin{tikzcd} K \arrow[r] & L \\ A \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

OK, now let's introduce the "variable" $y = x^{-1}$. Set $A' = \mathbf{C}[y] \subset K$. Let B' be the integral closure of A' in L. By the result of Jarod's lecture this is a finite ring extension of A'. We can also consider $A'' = \mathbf{C}[x, y] = \mathbf{C}[x, x^{-1}] \subset K$ and its integral closure $B'' \subset L$. This produces the following diagram

$$\begin{tikzcd} B' \arrow[r] & B'' & B \arrow[l] \\ A' \arrow[u] \arrow[r] & A'' \arrow[u] & A \arrow[u] \arrow[l] \end{tikzcd}$$

Having said all of this prove that $B'' = B'_y = B_x$. See Section 5 for the notation R_f. (You may use that the situation is symmetric in x and y and hence that you only need to prove either $B'' = B'_y$ or that $B'' = B_x$.)

Exercise 12.4. Notation as in Exercise 12.3. Let $f \in L$ be an element which is contained in B' and in B''. Show that $f \in \mathbf{C}$ using the following steps:

(1) Show that $B \cong A^{\oplus d}$ as an A-module. (Quote a theorem on modules over the polynomial ring $\mathbf{C}[x]$.)

(2) Show that $B' \cong (A')^{\oplus d}$ as an A'-module. (Quote a theorem on modules over polynomial ring $\mathbf{C}[y]$. Yes, this is silly.)
(3) Let $P(T) \in A[T]$ be the characteristic polynomial of multiplication by f on B as an A-linear map. Let $P'(T) \in A'[T]$ be the characteristic polynomial of multiplication by f on B' as an A'-linear map. Show that $P(T) = P'(T)$ in $A''[T]$. [Hint: Char Pol independent of chosen basis.]
(4) Conclude P has constant coefficients.

Exercise 12.5. Notation as in Exercise 12.3. Show that the group of units B^* sits in a short exact sequence

$$0 \to \mathbf{C}^* \to B^* \to B^*/\mathbf{C}^* \to 0$$

of abelian groups and that the group $B^*/\mathbf{C}^*$ on the right is a finitely generated free abelian group. [[[Hints: This exercise is a quite a bit harder... First of all, by symmetry it is OK to switch the roles of B and B' (this is just a notational convenience). Let $d = [L : K]$ as in Exercise 12.4. Suppose that $u_1, \ldots, u_{d+1}$ are units of B'. Try to find integers $m_1, \ldots, m_r \in \mathbf{Z}$ not all zero such that

$$u = u_1^{m_1} \ldots u_{d+1}^{m_{d+1}}$$

is also in B. Having found these then by Exercise 12.4 we see $u \in \mathbf{C}^*$. To find the m_i use the result from Exercise 12.2 and the result from the lectures that says that $A \subset B$ matches with $\hat{A} \subset \hat{B}$ in some sense, but try to be somewhat precise here. If you do not remember the statement ask me.]]]

100. Review Course Material

Review of material in course.

(1) Projective space $\mathbf{P}^n = (\mathbf{C}^{n+1} \setminus \{0\})/\mathbf{C}^*$ is defined as the set of nonzero vectors in $\mathbf{C}^{n+1}$ up to scaling.
(2) A point of $\mathbf{P}^n$ is denoted $[a_0 : a_1 : \ldots : a_n]$ which means that $(a_0, \ldots, a_n) \in \mathbf{C}^{n+1}$ is a nonzero vector and $[a_0 : a_1 : \ldots : a_n]$ is the corresponding point.
(3) For $F \in \mathbf{C}[X_0, \ldots, X_n]$ we defined

$$V_+(F) = \{[a_0 : a_1 : \ldots : a_n] \in \mathbf{P}^n \mid F(a_0, \ldots, a_n) = 0\}$$

and we showed that this is well defined.
(4) The standard affine opens U_i, $i = 0, \ldots, n$. We have

$$\mathbf{P}^n = U_0 \cup \ldots \cup U_n$$

where $U_i = \mathbf{P}^n \setminus V_+(X_i)$ is the set of points whose ith coordinate is nonzero. For each i we have a bijection

$$\Phi_i : U_i \longrightarrow \mathbf{C}^n, \quad [a_0 : a_1 : \ldots : a_n] \longmapsto \left(\frac{a_0}{a_i}, \ldots, \widehat{\frac{a_0}{a_i}}, \ldots, \frac{a_n}{a_i}\right)$$

whose inverse is given by the map

$$(c_1, \ldots, c_n) \longmapsto [c_1 : \ldots : c_{i-1} : 1 : c_i : \ldots : c_n].$$

(5) Let $\pi : \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{P}^n$ denote the map $(a_0, \ldots, a_n) \mapsto [a_0 : a_1 : \ldots : a_n]$.
(6) There is a topology on $\mathbf{P}^n$ defined by saying $U \subset \mathbf{P}^n$ is open $\Leftrightarrow \pi^{-1}(U) \subset \mathbf{C}^{n+1} \setminus \{0\}$ is open in the usual topology. This will be called the *usual topology*.

(7) The map π is continuous and open (in the usual topologies). It follows that $\mathbf{P}^n$ is Hausdorff in the usual topology.
(8) The maps Φ_i are homeomorphisms in the usual topologies and the standard affine opens are open. (This determines the usual topology.)
(9) The unit sphere $S^{2n+2} \subset \mathbf{C}^{n+1} \setminus \{0\}$ (the set of points $(a_0, \ldots a_n)$ such that $\sum |a_i|^2 = 1$) surjects onto $\mathbf{P}^n$ we see that $\mathbf{P}^n$ is compact in the usual topology.
(10) The space $\mathbf{P}^n$ is a compact topological manifold in the usual topology.
(11) We defined a topology on $\mathbf{P}^n$ whose closed subsets are

$$Z = \bigcap\nolimits_{F \in E} V_+(F)$$

where $E \subset \mathbf{C}[X_0, \ldots, X_n]$ is a subset consisting of homogeneous elements. This is the *Zariski topology* on $\mathbf{P}^n$.
(12) A Zariski closed subset of $\mathbf{P}^n$ is usual closed.
(13) The maps $\Phi_i : U_i \to \mathbf{C}^n$ are homeomorphisms in the Zariski topologies and the $U_i \subset \mathbf{P}^n$ are open in the Zariski topology. (This determines the Zariski topology.)
(14) The Zariski topological space $\mathbf{P}^n$ is Noetherian.
(15) A *quasi-projective variety* is a Zariski irreducible locally closed subset $X \subset \mathbf{P}^n$ for some n.
(16) A *projective variety* is a Zariski irreducible closed subset $X \subset \mathbf{P}^n$ for some n.
(17) The maps

$$\Phi_j \circ \Phi_i^{-1}|_{\Phi_i(U_i \cap U_j)} : \Phi_i(U_i \cap U_j) \longrightarrow \Phi_j(U_i \cap U_j)$$

are isomorphisms of q-affine varieties.
(18) For any quasi-projective variety $X \subset \mathbf{P}^n$ with $X \subset U_i$ and $X \subset U_j$ the images $\Phi_i(X)$ and $\Phi_j(X)$ are quasi-affine varieties (by the above) and

$$\Phi_j \circ \Phi_i^{-1}|_{\Phi_i(X)} : \Phi_i(X) \longrightarrow \Phi_j(X)$$

is an isomorphisms of quasi-affine varieties.
(19) If $X \subset \mathbf{P}^n$ is a quasi-projective variety with $X \subset U_i$ for some i the we define the algebra of *regular functions on* X by the rule

$$\mathcal{O}(X) = \{f : X \to \mathbf{C} \mid \begin{array}{c}\text{the map } \Phi_i(X) \to \mathbf{C}, \Phi_i(x) \mapsto f(x) \text{ is a regular} \\ \text{function on the quasi-affine variety } \Phi_i(X)\end{array}\}$$

This is independent of the choice of i such that $X \subset U_i$ by (18).
(20) If $X \subset \mathbf{P}^n$ is a quasi-projective variety we define the algebra of *regular functions on* X by the rule

$$\mathcal{O}(X) = \{f : X \to \mathbf{C} \mid f|_{X \cap U_i} \in \mathcal{O}(X \cap U_i) \text{ for } i = 0, \ldots, n\}$$

This makes sense because we have defined $\mathcal{O}(X \cap U_i)$ in (19).
(21) A regular function f on a quasi-projective variety $X \subset \mathbf{P}^n$ is continuous in both the Zariski and the usual topologies. This is true because we have seen this holds for $f|_{X \cap U_i}$.
(22) If X is a quasi-projective variety and $Y \subset X$ is a subvariety, then the restriction $f|_Y$ of a regular function on X is a regular function on Y. (This is true because we've seen it holds on $X \cap U_i$.)

(23) Let $X \subset \mathbf{P}^n$ be a quasi-projective variety. Let $f : X \to \mathbf{C}$ be a map of sets. The following are equivalent
  (a) f is a regular function, and
  (b) there exists an open covering $X = V_1 \cup V_2 \cup \ldots \cup V_m$ such that each restriction $f|_{V_j}$ is a regular function.

(24) A *morphism* $\varphi : X \to Y$ of quasi-projective varieties is a continuous map (in Zariski topology) such that for every open $V \subset Y$ and any regular function $f \in \mathcal{O}(V)$ on V the composition $f \circ \varphi|_{\varphi^{-1}(V)} : \varphi^{-1}(V) \to \mathbf{C}$ is a regular function on $\varphi^{-1}(V)$.

(25) If $X \subset \mathbf{P}^n$ and $Y \subset \mathbf{P}^m$ are quasi-projective varieties and $\varphi : X \to Y$ is a map of sets then the following are equivalent:
  (a) φ is a morphism,
  (b) φ composed with the map $Y \to \mathbf{P}^m$ is a morphism from X to $\mathbf{P}^m$,
  (c) there exists an open covering $X = V_1 \cup V_2 \cup \ldots \cup V_m$ such that each restriction $f|_{V_i}$ is a morphism,
  (d) for each $j \in \{0, \ldots, m\}$ we have that $\varphi^{-1}(U_j)$ is open in X, and the composition

$$\begin{tikzcd} \varphi^{-1}(U_j) \arrow[r, "\varphi|_{\varphi^{-1}(U_j)}"'] \arrow[rr, bend left] & U_j \arrow[r, "\Phi_j"'] & \mathbf{C}^m \end{tikzcd}$$

  is a morphism, and
  (e) there exists an open covering $X = \bigcup V_i$ such that for each i you have $\varphi(V_i) \subset U_{j(i)}$ and moreover the composition

$$\begin{tikzcd} V_i \arrow[r, "\varphi|_{V_i}"'] \arrow[rr, bend right] & U_{j(i)} \arrow[r, "\Phi_{j(i)}"'] & \mathbf{C}^m \end{tikzcd}$$

  is a morphism.

The easiest way to use these is to find V_i as in the last condition such that moreover each V_i is *also* contained in a standard affine open of $\mathbf{P}^n$, since in that case you reduced to checking that the restriction is a morphism of quasi-affine varieties .