Commutative Algebra

Exercises 11

Let A be a ring. Recall that a *finite locally free* A-module M is a module such that for every $\mathfrak{p} \in \operatorname{Spec} A$ there exists an $f \in A$, $f \notin \mathfrak{p}$ such that M_f is a finite free A_f-module.

1. Let A be a ring.
(a) Suppose that M is a finite locally free A-module, and suppose that $\varphi : M \to M$ is an endomorphism. Define/construct the *trace* and *determinant* of φ and prove that your construction is "functorial in the triple (A, M, φ)".
(b) Show that if M, N are finite locally free A-modules, and if $\varphi : M \to N$ and $\psi : N \to M$ then $Trace(\varphi \circ \psi) = Trace(\psi \circ \varphi)$ and $Det(\varphi \circ \psi) = Det(\psi \circ \varphi)$.
(c) In case M is finite locally free show that Det defines a multiplicative map $End_A(M) \to A$.

2. Now suppose that B is an A-algebra which is finite locally free as an A-module, in other words B is a finite locally free A-algebra.
(a) Define $Trace_{B/A}$ and $Norm_{B/A}$ using $Trace$ and Det as defined above.
(b) Let $b \in B$ and let $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced morphism. Show that $\pi(V(b)) = V(Norm_{B/A}(b))$. (Recall that $V(f) = \{\mathfrak{p} \mid f \in \mathfrak{p}\}$.)
(c) (Base change.) Suppose that $i : A \to A'$ is a ring map. Set $B' = B \otimes_A A'$. Indicate why $i(Norm_{B/A}(b))$ equals $Norm_{B'/A'}(b \otimes 1)$.
(d) Compute $Norm_{B/A}(b)$ when $B = A \times A \times A \times \ldots \times A$ and $b = (a_1, \ldots, a_n)$.
(e) Compute the norm of $y - y^3$ under the finite flat map $\mathbb{Q}[x] \to \mathbb{Q}[y]$, $x \to y^n$. (Hint: use the "base change" $A = \mathbb{Q}[x] \subset A' = \mathbb{Q}(\zeta_n)(x^{1/n})$.)

Remark. Let $h \in \mathbb{Z}[y]$ be a monic polynomial of degree d. Then:
(a) The map $A = \mathbb{Z}[x] \to B = \mathbb{Z}[y]$, $x \mapsto h$ is finite locally free of rank d.
(b) For all primes p the map $A_p = \mathbb{F}_p[x] \to B_p = \mathbb{F}_p[y]$, $y \mapsto h(y) \bmod p$ is finite locally free of rank d.

3. Let h, A, B, A_p, B_p be as in the remark. For $f \in \mathbb{Z}[x, u]$ we define $f_p(x) = f(x, x^p) \bmod p \in \mathbb{F}_p[x]$. For $g \in \mathbb{Z}[y, v]$ we define $g_p(y) = g(y, y^p) \bmod p \in \mathbb{F}_p[y]$.
(a) Give an example of a h and g such that there does not exist a f with the property

$$f_p = Norm_{B_p/A_p}(g_p).$$

(b) Show that for any choice of h and g as above there exists a nonzero f such that for all p we have

$$Norm_{B_p/A_p}(g_p) \text{ divides } f_p.$$

If you want you can restrict to the case $h = y^n$, even with $n = 2$, but it is true in general.
(c) Discuss the relevance of this to Exercises 6 & 7 of the previous set.

4. Unsolved problems. They may be really hard or they may be easy. I don't know.
(a) Is there any $f \in \mathbb{Z}[x, u]$ such that f_p is irreducible for an inifinite number of p?
(b) Let $f \in \mathbb{Z}[x, u]$ nonzero, and suppose $\deg_x(f_p) = dp + d'$ for all large p. (In other words $\deg_u(f) = d$ and the coefficient c of u^d in f has $\deg_x(c) = d'$.) Suppose we can write $d = d_1 + d_2$ and $d' = d'_1 + d'_2$ with $d_1, d_2 > 0$ and $d'_1, d'_2 \geq 0$ such that for all sufficiently large p there exists a factorization

$$f_p = f_{1,p} f_{2,p}$$

with $\deg_x(f_{1,p}) = d_1 p + d'_1$. Is it true that f comes about via a norm construction as in Exercise 4? (More precisely, are there a h and g such that $Norm_{B_p/A_p}(g_p)$ divides f_p for all $p >> 0$.)
(c) Analogous question to the one in (b) but now with $f \in \mathbb{Z}[x_1, x_2, u_1, u_2]$ irreducible and just assuming that $f_p(x_1, x_2) = f(x_1, x_2, x_1^p, x_2^p) \bmod p$ factors for all $p >> 0$.