Commutative Algebra

Excercises 4

Let $\phi : A \to B$ be a homomoprhism of rings. We say that the *going-up theorem* holds for ϕ if the following condition is satisfied:

(GU) for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $P \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$, there exists $P' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$ such that $P \subset P'$.

Similarly, we say that the *going-down theorem* holds for ϕ if the following condition is satisfied:

(GD) for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $P' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$, there exists $P \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$ such that $P \subset P'$.

1. In each of the following cases determine whether (GU), (GD) holds, and explain why. (Use any Prop/Thm/Lemma you can find, but check the hypotheses in each case.)

(a) k is a field, $A = k$, $B = k[x]$.
(b) k is a field, $A = k[x]$, $B = k[x, y]$.
(c) $A = \mathbb{Z}$, $B = \mathbb{Z}[1/11]$.
(d) k is an algebraically closed field, $A = k[x, y]$, $B = k[x, y, z]/(x^2 - y, z^2 - x)$.
(e) $A = \mathbb{Z}$, $B = \mathbb{Z}[i, 1/(2 + i)]$.
(f) $A = \mathbb{Z}$, $B = \mathbb{Z}[i, 1/(14 + 7i)]$.
(g) k is an algebraically closed field, $A = k[x]$, $B = k[x, y, 1/(xy - 1)]/(y^2 - y)$.

2. Let k be an algebraically closed field. Compute the image in $\mathrm{Spec}(k[x, y])$ of the following maps:

(a) $\mathrm{Spec}(k[x, yx^{-1}]) \to \mathrm{Spec}(k[x, y])$, where $k[x, y] \subset k[x, yx^{-1}] \subset k[x, y, x^{-1}]$. (Hint: To avoid confusion, give the element yx^{-1} another name.)
(b) $\mathrm{Spec}(k[x, y, a, b]/(ax - by - 1)) \to \mathrm{Spec}(k[x, y])$.
(c) $\mathrm{Spec}(k[t, 1/(t - 1)]) \to \mathrm{Spec}(k[x, y])$, induced by $x \mapsto t^2$, and $y \mapsto t^3$.
(d) $k = \mathbb{C}$ (complex numbers), $\mathrm{Spec}(k[s, t]/(s^3 + t^3 - 1)) \to \mathrm{Spec}(k[x, y])$, where $x \mapsto s^2$, $y \mapsto t^2$.

Remark. Finding the image as above usually is done by using elimination theory.

3. Let k be a field. Show that the following pairs of k-algebras are not isomorphic:

(a) $k[x_1, \ldots, x_n]$ and $k[x_1, \ldots, x_{n+1}]$ for any $n \geq 1$.
(b) $k[a, b, c, d, e, f]/(ab + cd + ef)$ and $k[x_1, \ldots, x_n]$ for $n = 5$.
(c) $k[a, b, c, d, e, f]/(ab + cd + ef)$ and $k[x_1, \ldots, x_n]$ for $n = 6$.

Remark. Of course the idea of this exercise is to find a simple argument in each case rather than applying a "big" theorem. Nonetheless it is good to be guided by general principles.