Commutative Algebra

Excercises 8

Let R be a graded ring. A *homogeneous* ideal is simply an ideal $I \subset R$ which is also a graded submodule of R. Equivalently, it is an ideal generated by homogeneous elements. Equivalently, if $f \in I$ and

$$f = f_0 + f_1 + \ldots + f_n$$

is the decomposition of f into homogenous pieces in R then $f_i \in I$ for each i. We define $\mathrm{Proj}(R)$ to be the set of homogenous, prime ideals $\mathfrak{p}$ of R such that $R_+ \not\subset \mathfrak{p}$. Note that $\mathrm{Proj}(R)$ is a subset of $\mathrm{Spec}(R)$ and hence has a natural induced topology.

Let $R = \oplus_{d \geq 0} R_d$ be a graded ring, let $f \in R_d$ and assume that $d \geq 1$. We define $R_{(f)}$ to be the subring of R_f consisting of elements of the form r/f^n with r homogenous and $\deg(r) = nd$. Furthermore, we define

$$D_+(f) = \{\mathfrak{p} \in \mathrm{Proj}(R) | f \notin \mathfrak{p}\}.$$

Finally, for a homogenous ideal $I \subset R$ we define $V_+(I) = V(I) \cap \mathrm{Proj}(R)$.

1. Topology on $\mathrm{Proj}(R)$. With notations as above:

(a) Show that $D_+(f)$ is open in $\mathrm{Proj}(R)$, show that $D_+(ff') = D_+(f) \cap D_+(f')$.

(b) Let $g = g_0 + \ldots + g_m$ be an element of R with $g_i \in R_i$. Express $D(g) \cap \mathrm{Proj}(R)$ in terms of $D_+(g_i)$, $i \geq 1$ and $D(g_0) \cap \mathrm{Proj}(R)$. No proof necessary.

(c) Let $g \in R_0$ be a homogenous element of degree 0. Express $D(g) \cap \mathrm{Proj}(R)$ in terms of $D_+(f_\alpha)$ for a suitable family $f_\alpha \in R$ of homogenous elements of positive degree.

(d) Show that the collection $\{D_+(f)\}$ of opens forms a basis for the topology of $\mathrm{Proj}(R)$.

(e) Show that there is a canonical bijection $D_+(f) \to \mathrm{Spec}(R_{(f)})$.

(f) Show that the map from (e) is a homeomorphism.

(g) Give an example of an R such that $\mathrm{Proj}(R)$ is not quasi-compact. No proof necessary.

(i) Show that any closed subset $T \subset \mathrm{Proj}(R)$ is of the form $V_+(I)$ for some homogenous ideal $I \subset R$.

Remark. There is a continuous map $\mathrm{Proj}(R) \longrightarrow \mathrm{Spec}(R_0)$.

2. If $R = A[X]$ with $\deg(X) = 1$, show that the natural map $\mathrm{Proj}(R) \to \mathrm{Spec}(A)$ is a bijection and in fact a homeomorphism.

3. Blowing up: part I. In this exercise $R = Bl_I(A) = A \oplus I \oplus I^2 \oplus \ldots$. Consider the natural map $b : \mathrm{Proj}(R) \to \mathrm{Spec}(A)$. Set $U = \mathrm{Spec}(A) - V(I)$. Show that

$$b : b^{-1}(U) \longrightarrow U$$

is a homeomorphism.

Thus we may think of U as an open subset of $\mathrm{Proj}(R)$. Let $Z \subset \mathrm{Spec}(A)$ be an irreducible closed subscheme with generic point $\xi \in Z$. Assume that $\xi \notin V(I)$, in other words $Z \not\subset V(I)$, in other words $\xi \in U$, in other words $Z \cap U \neq \emptyset$. We define the *strict transform*

Z' of Z to be the closure of the unique point ξ' lying above ξ. Another way to say this is that Z' is the closure in $\text{Proj}(R)$ of the locally closed subset $Z \cap U \subset U \subset \text{Proj}(R)$.

4. Blowing up: Part II. Let $A = k[x, y]$ where k is a field, and let $I = (x, y)$. Let R be the blow up algebra for A and I.

(a) Show that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{y\})$ are disjoint.
(b) Show that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{x - y^2\})$ are not disjoint.
(c) Find an ideal $J \subset A$ such that $V(J) = V(I)$ and such that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{x - y^2\})$ are disjoint.

5. Let R be a graded ring.

(a) Show that $\text{Proj}(R)$ is empty if $R_n = (0)$ for all $n >> 0$.
(b) Show that $\text{Proj}(R)$ is an irreducible topological space if R is a domain and R_+ is not zero. (Recall that the empty topological space is not irreducible.)

6. Blowing up: Part III. Consider A, I and U, Z as in the definition of strict transform. Let $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. Let $\bar{A} = A/\mathfrak{p}$ and let $\bar{I}$ be the image of I in $\bar{A}$.

(a) Show that there exists a surjective ring map $R := Bl_I(A) \to \bar{R} := Bl_{\bar{I}}(\bar{A})$.
(b) Show that the ring map above induces a bijective map from $\text{Proj}(\bar{R})$ onto the strict transform Z' of Z. (This is not so easy. Hint: Use 5(b) above.)
(c) Conclude that the strict transform $Z' = V_+(P)$ where $P \subset R$ is the homogenous ideal defined by $P_d = I^d \cap \mathfrak{p}$.
(d) Suppose that $Z_1 = V(\mathfrak{p})$ and $Z_2 = V(\mathfrak{q})$ are irreducible closed subsets defined by prime ideals such that $Z_1 \not\subset Z_2$, and $Z_2 \not\subset Z_1$. Show that blowing up the ideal $I = \mathfrak{p} + \mathfrak{q}$ separates the strict transforms of Z_1 and Z_2, i.e., $Z_1' \cap Z_2' = \emptyset$. (Hint: Consider the homogenous ideal P and Q from part (c) and consider $V(P + Q)$.)