# Commutative Algebra

Excercises 9

A Noetherian local ring $A$ is said to be *Cohen-Macauley* of dimension $d$ if it has dimension $d$ and there exists a system of parameters $x_1, \ldots, x_d$ for $A$ such that $x_i$ is a nonzero divisor in $A/(x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, d$.

**1.** Cohen-Macauley rings of dimension 1. Part I: Theory.

**(a)** Let $(A, \mathfrak{m})$ be a local Noetherian with $\dim A = 1$. Show that if $x \in \mathfrak{m}$ is not a zero divisor then

- **(1)** $\dim A/xA = 0$, in other words $A/xA$ is Artinian, in other words $\{x\}$ is a system of parameters for $A$.
- **(2)** $A$ is has no embedded prime.

**(b)** Conversely, let $(A, \mathfrak{m})$ be a local Noetherian ring of dimension 1. Show that if $A$ has no embedded prime then there exists a nonzero divisor in $\mathfrak{m}$.

**2.** Cohen-Macauley rings of dimension 1. Part II: Examples.

**b.** Let $A$ be the local ring at $(x, y)$ of $k[x, y]/(x^2, xy)$.

- **(1)** Show that $A$ has dimension 1.
- **(2)** Prove that every element of $\mathfrak{m} \subset A$ is a zero divisor.
- **(3)** Find $z \in \mathfrak{m}$ such that $\dim A/zA = 0$ (no proof required).

**c.** Let $A$ be the local ring at $(x, y)$ of $k[x, y]/(x^2)$. Find a nonzero divisor in $\mathfrak{m}$ (no proof required).

**3.** Local rings of embedding dimension 1. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring of embedding dimension 1, i.e.,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1.$$

Show that the function $f(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is either constant with value 1, or its values are

$$1, 1, \ldots, 1, 0, 0, 0, 0, 0, \ldots$$

**4.** Regular local rings of dimension 1. Suppose that $(A, \mathfrak{m}, k)$ is a regular Noetherian local ring of dimension 1. Recall that this means that $A$ has dimension 1 and embedding dimension 1, i.e.,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1.$$

Let $x \in \mathfrak{m}$ be any element whose class in $\mathfrak{m}/\mathfrak{m}^2$ is not zero.

**(a)** Show that for every element $y$ of $\mathfrak{m}$ there exists an integer $n$ such that $y$ can be written as $y = ux^n$ with $u \in A^*$ a unit.

**(b)** Show that $x$ is a nonzero divisor in $A$.

**(c)** Conclude that $A$ is a domain.

**5.** Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring with associated graded $Gr_{\mathfrak{m}}(A)$.

**(a)** Suppose that $x \in \mathfrak{m}^d$ maps to a nonzero divisor $\bar{x} \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ in degree $d$ of $Gr_{\mathfrak{m}}(A)$. Show that $x$ is a nonzero divisor.

**(b)** Suppose the depth of $A$ is at least 1. Namely, suppose that there exists a nonzero divisor $y \in \mathfrak{m}$. In this case we can do better: assume just that $x \in \mathfrak{m}^d$ maps to the element $\bar{x} \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ in degree $d$ of $Gr_{\mathfrak{m}}(A)$ which is a nonzero divisor on sufficiently high degrees: $\exists N$ such that for all $n \geq N$ the map

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \xrightarrow{\bar{x}\cdot} \mathfrak{m}^{n+d}/\mathfrak{m}^{n+d+1}$$

is injective. Then show that $x$ is a nonzero divisor.

**6.** Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring of dimension 1. Assume also that the embedding dimension of $A$ is 2, i.e., assume that

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2.$$

Notation: $f(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Pick generators $x, y \in \mathfrak{m}$ and write $Gr_{\mathfrak{m}}(A) = k[\bar{x}, \bar{y}]/I$ for some homogenous ideal $I$.

**(a)** Show that there exists a homogenous element $F \in k[\bar{x}, \bar{y}]$ such that $I \subset (F)$ with equality in all sufficiently high degrees.
**(b)** Show that $f(n) \leq n + 1$.
**(c)** Show that if $f(n) < n + 1$ then $n \geq \deg(F)$.
**(d)** Show that if $f(n) < n + 1$, then $f(n + 1) \leq f(n)$.
**(e)** Show that $f(n) = \deg(F)$ for all $n >> 0$.

**7.** Cohen-Macauley rings of dimension 1 and embedding dimension 2. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring which is Cohen-Macauley of dimension 1. Assume also that the embedding dimension of $A$ is 2, i.e., assume that

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2.$$

Notations: $f$, $F$, $x, y \in \mathfrak{m}$, $I$ as in Ex. 6 above. Please use any results from the problems above.

**(a)** Suppose that $z \in \mathfrak{m}$ is an element whose class in $\mathfrak{m}/\mathfrak{m}^2$ is a linear form $\alpha\bar{x} + \beta\bar{y} \in k[\bar{x}, \bar{y}]$ which is coprime with $f$.
  **(1)** Show that $z$ is a nonzero divisor on $A$.
  **(2)** Let $d = \deg(F)$. Show that $\mathfrak{m}^n = z^{n+1-d}\mathfrak{m}^{d-1}$ for all sufficiently large $n$. (Hint: First show $z^{n+1-d}\mathfrak{m}^{d-1} \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is surjective by what you know about $Gr_{\mathfrak{m}}(A)$. Then use NAK.)
**(b)** What condition on $k$ garantees the existence of such a $z$? (No proof required; it's too easy.)

Now we are going to assume there exists a $z$ as above. This turns out to be a harmless assumption (in the sense that you can reduce to the situation where it holds in order to obtain the results in parts (d) and (e) below).

**(c)** Now show that $\mathfrak{m}^\ell = z^{\ell-d+1}\mathfrak{m}^{d-1}$ for all $\ell \geq d$.
**(d)** Conclude that $I = (F)$.
**(e)** Conclude that the function $f$ has values

$$2, 3, 4, \ldots, d - 1, d, d, d, d, d, d, d, \ldots$$

This suggests that a local Noetherian Cohen-Macauley ring of dimension 1 and embedding dimension 2 is of the form $B/FB$, where $B$ is a 2-dimensional regular local ring. This is more or less true (under suitable "niceness" properties of the ring).