TRANSCENDENCE DEGREE GIVES A DIMENSION FUNCTION

Throughout k is a field. For any finite type k-algebra A we consider the function

$$\delta : \mathrm{Spec}(A) \longrightarrow \mathbf{Z}, \quad \mathfrak{p} \longmapsto \delta(\mathfrak{p}) = \mathrm{trdeg}_k\big(\kappa(\mathfrak{p})\big)$$

Lemma 0.1. *Let $A \subset B$ be an integral extension of domains. If $J \subset B$ is a nonzero ideal, then $A \cap J$ is nonzero.*

Proof. Let $b \in J$ be nonzero. Let $P = t^d + a_1 t^{d-1} + \ldots + a_d \in A[t]$ be a monic polynomial of minial degree with $P(b) = 0$. Then a_d is nonzero, since otherwise we may replace P by P/t (here we use that B is a domain and b is nonzero) and lower the degree of P. Then $a_d = -b^d - a_1 b^{d-1} - \ldots - a_{d-1} b \in J \cap A$. □

Lemma 0.2. *Let $A \subset B$ be an extension of domains and denote $K \subset L$ their fraction fields. Let $b \in B$ be integral over A. If A is normal, then the minimal polynomial $P \in K[t]$ of b viewed as an element of L has coefficients in A.*

Proof. Let $Q \in A[t]$ be a monic polynomial such that $Q(b) = 0$[1]. Since P is the minimal polynomial of b we see that P is monic and P divides Q in $K[t]$. Choose a splitting field M/L of Q. Write

$$Q = (t - \beta_1) \ldots (t - \beta_m)$$

with $\beta_1, \ldots, \beta_m$ the roots of Q in M. Clearly, $\beta_1, \ldots, \beta_m$ are integral over A. Since $P|Q$ and P is monic, we see that after a permutation we can find $n \leq m$ such that

$$P = (t - \beta_1) \ldots (t - \beta_n)$$

Thus the coefficients are polynomials in the β_j's and in particular they are integral over A. Since they are also in K and A is normal, we see that they are in A. □

Lemma 0.3. *Let A be a finite type k-algebra. For $g \in A$ there is a finite type k-algebra $A_g = A[t]/(gt-1)$ with the following property: the induced map $\mathrm{Spec}(A_g) \to \mathrm{Spec}(A)$ induces a homeomorphism onto $D(g) \subset \mathrm{Spec}(A)$ and identifies residue fields.*

Proof. Omitted. □

Lemma 0.4. *Let A be of finite type over a field k. Let $\mathfrak{q} \subset \mathfrak{p} \subset A$ be distinct primes. Then $\delta(\mathfrak{q}) > \delta(\mathfrak{p})$.*

Proof. Set $A' = A/\mathfrak{q}$, $\mathfrak{q}' = (0)$, and $\mathfrak{p}' = \mathfrak{p}/\mathfrak{q}$. Then $\kappa(\mathfrak{p}') = \kappa(\mathfrak{p})$ and $\kappa(\mathfrak{q}') = \kappa(\mathfrak{q})$; please prove this yourselves. Thus we may assume A is a domain and $\mathfrak{q} = (0)$. In other words, we have to show: if A is a domain and $\mathfrak{p} \subset A$ is a nonzero prime, then the transcendence degree of $\kappa(\mathfrak{p})$ over k is less than the transcendence degree of $f.f.(A)$.

Choose $P = k[x_1, \ldots, x_r] \subset A$ finite (Noether normalization, see previous lecture). Then

$$k(x_1, \ldots, x_r) \subset f.f.(A)$$

[1]The existence of Q guarantees that b viewed as an element of L is indeed algebraic over K and hence the statement of the lemma makes sense.

is a finite extension, hence the transcendence degree of $f.f.(A)$ is r. By Lemma 0.1 there is a nonzero element $g \in k[x_1, \ldots, x_r]$ contained in $\mathfrak{p}$. Thus the images $\overline{x}_1, \ldots, \overline{x}_r$ of the elements $x_1, \ldots, x_r$ in $\kappa(\mathfrak{p}) = f.f.(A/\mathfrak{p})$ aren't algebraically independent. Now $\kappa(\mathfrak{p})$ is algebraic over the subfield generated by k and $\overline{x}_1, \ldots, \overline{x}_r$; please prove this yourselves. We conclude that the transcendence degree of $\kappa(\mathfrak{p})$ over k is strictly less than r. □

Lemma 0.5. *Let A be of finite type over a field k. Let $\mathfrak{q} \subset \mathfrak{p} \subset A$ be distinct prime ideals with no prime ideal strictly in between. Then $\delta(\mathfrak{p}) + 1 = \delta(\mathfrak{q})$.*

Proof. After replacing A by $A/\mathfrak{q}$ as in the proof of Lemma 0.4 we may assume $\mathfrak{q} = (0)$ and hence $\mathfrak{p} \subset A$ is a prime ideal minimal with the property of not being zero. Our goal is to show that $\text{trdeg}_k \kappa(\mathfrak{p}) + 1 = \text{trdeg}_k f.f.(A)$.

Pick nonzero f in $\mathfrak{p}$. Then $\mathfrak{p}$ is minimal over (f), i.e., $\mathfrak{p}$ defines a generic point of $V(f)$. By a previous result $V(f)$ has a finite number of generic points besides $\mathfrak{p}$. By prime avoidance can pick g in those primes but not in $\mathfrak{p}$. After replacing A by A_g we get $\mathfrak{p} = \sqrt{(f)}$. Some details omitted; see Lemma 0.3 to see why it is permissible to do this replacement.

Choose $P = k[x_1, \ldots, x_r] \subset A$ finite (Noether normalization). Then $r = \text{trdeg}_k f.f.(A)$, see proof of Lemma 0.4. For all elements of A the minimal polynomial has coefficients in P by Lemma 0.2 and the normality of the polynomial algebra. In particular get $Nm : A \to P$ because the norm of an element is a power of the last coefficient of the minimal polynomial (see previous lecture).

Let $\mathfrak{q} = P \cap \mathfrak{p}$. Then $g \in \mathfrak{q}$ implies $g^n = af$ for some $n > 0$ and $a \in A$. Let d be the degree of the extension $f.f.(A)/f.f.(P)$. Then

$$g^{nd} = Nm(g)^n = Nm(g^n) = Nm(a)Nm(f)$$

by properties of the norm (see previous lecture).

We have $Nm(f) \in \mathfrak{q}$: the last coefficient of its minimal polynomial is in $\mathfrak{p}$ and in P and $Nm(f)$ is a power of it (compare with the proof of Lemma 0.1). Thus, we see that one of the irreducible factors, say g, of $Nm(f)$ is in $\mathfrak{q}$. Applying the displayed equation above we see that $Nm(f)$ is a power of the irreducible g up to a unit.

We claim $\mathfrak{q} = (g)$. Namely, if $h \in \mathfrak{q}$, then applying the displayed equation for h we see that g divides a power of h, hence $h \in (g)$.

The transcendence degree of $\kappa(\mathfrak{q})$ over k is $r-1$ (see previous lecture). The extension $\kappa(\mathfrak{q}) \subset \kappa(\mathfrak{p})$ is finite because $P \subset A$ is finite; please prove this yourselves. Thus the transcendence degree of $\kappa(\mathfrak{p})$ over k is $r - 1$ as well. □

Lemma 0.6 (Hilbert Nullstellensatz)**.** *Let A be finite type over a field k. Let $\mathfrak{p} \subset A$ be a prime. Then the following are equivalent*

1. $\mathfrak{p}$ *is a maximal ideal,*
2. *trdeg*$_k \kappa(\mathfrak{p}) = 0$,
3. $\kappa(\mathfrak{p})$ *is finite over* k.

Proof. After replacing A by $A/\mathfrak{p}$ we see that we have to show the following: given a domain A of finite type over k the following are equivalent

1. A is a field,
2. $\text{trdeg}_k f.f.(A) = 0$,

(3) $f.f.(A)$ is finite over k.

The equivalence of (2) and (3) is immediate from our discussion of transcendence degree of finitely generated field extensions of k in a previous lecture.

(2) $\Rightarrow$ (1): If A is not a field, then A has a nonzero prime ideal. Then Lemma 0.4 immediately implies $\text{trdeg}_k f.f.(A) > 0$.

(1) $\Rightarrow$ (2). Assume $r = \text{trdeg}_k f.f.(A) > 0$. Then there is a finite inclusion $P = k[x_1, \ldots, x_r] \subset A$ by Noether normalization (where $r = r$, see proof of Lemma 0.4). Then x_1 cannot be invertible in A, since if so, then $x_1^{-1} \in P$ (which is absurd) by Lemma 0.2 and the normality of the polynomial algebra P. Thus A is not a field because x_1 is not invertible in A. □

Theorem 0.7. *Let A be a finite type algebra over a field k. The function*

$$\delta : \text{Spec}(A) \longrightarrow \mathbf{Z}, \quad \mathfrak{p} \longmapsto \delta(\mathfrak{p}) = trdeg_k\big(\kappa(\mathfrak{p})\big)$$

is a dimension function and we have $\delta(\mathfrak{p}) = 0$ if and only if $\mathfrak{p}$ is a closed point.

Proof. Combine Lemmas 0.4, 0.5, and 0.6. □

There are many things you can conclude from this. Let us give three examples.

Lemma 0.8. *Let A be a finite type k-algebra and set $X = \text{Spec}(A)$.*

(1) *If $x \in X$, then there is a specialization $x \rightsquigarrow y$ with y closed in X.*
(2) *We have $\dim(X) = \max\{\delta(x) \mid x \in X\}$.*
(3) *For $x \in X$ with $\delta(x) > 0$ the set of closed points of $Z = \overline{\{x\}}$ is infinite.*

Proof. The proof of (1) is formal from the theorem. Namely, if $\delta(x) = 0$, then we take $y = x$ (this is forced). If $\delta(x) > 0$ we see that x is not a closed point of X. Hence we can find $x \rightsquigarrow x'$ in X with $\delta(x) > \delta(x')$. If $\delta(x') > 0$, then we can do it again. Thus we can continue

$$x \rightsquigarrow x' \rightsquigarrow x'' \rightsquigarrow \ldots \rightsquigarrow x^{(e)}$$

until we hit a final point $x^{(e)} \in X$ with $\delta(x^{(e)}) = 0$. Set $y = x^{(e)}$. This is a closed point by the theorem. Since specialization is transitive we see that $x \rightsquigarrow y$.

Proof of (2). Since X is a sober[2] topological space the dimension of X is equal to the supremum of the lenghts of chains of nontrivial specializations

$$x_n \rightsquigarrow x_{n-1} \rightsquigarrow \ldots \rightsquigarrow x_0$$

in X. By the properties of a dimension function, we can assume $\delta(x_i) = \delta(x_{i-1}) + 1$. Since every prime in a ring contains a minimal prime and is contained in a maximal ideal, we may assume that x_n is a generic point of an irreducible component of X and x_0 a closed point. Then $n = \delta(x_n)$. Finally, there are only finitely many irreducible components of X hence the supremum is attained.

To prove (3) we use that X has a basis for its topology consisting of opens U which also have the property with respect to $\delta|_U$; namely the principal opens $D(g) = \text{Spec}(A_g)$, see Lemma 0.3. Suppose we have closed points $x_1, \ldots, x_n \in Z$. Then we can choose an open $U \subset X$ as above with $x \in U$ and $x_1 \notin U, \ldots, x_n \notin U$. By (1) applied to $x \in U$ we can find $x \rightsquigarrow y$ in U with y closed in U, equivalent $\delta(y) = 0$. Then $y \in X$ is closed because the function δ is the same for y viewed as a point

[2]This means that every closed irreducible subset Z is of the form $\overline{\{\eta\}}$ for a unique *generic point* $\eta \in Z$.

of U or of X! Since $y \in Z$ is not equal to $x_1, \ldots, x_n$ it is a "new" closed point as desired. $\square$

Example 0.9. For example, part (3) of the last lemma says that $k[x_1, \ldots, x_n]$ has infinitely many maximal ideals if $n > 0$. This is obvious if k is infinite, as you can take the ideals $(x_1 - a_1, \ldots, x_n - a_n)$ for $(a_1, \ldots, a_n) \in k^n$. But it is true even if k is a finite field. More interestingly perhaps, suppose $k = \mathbf{Q}$ and consider $A = \mathbf{Q}[x, y]/(x^2 + y^2 + 1)$. Then there are no *$Q$-rational points on* $\mathrm{Spec}(A)$, i.e., there are no maximal ideals $\mathfrak{m} \subset A$ with $\kappa(\mathfrak{m}) \cong \mathbf{Q}$. However, there are still infinitely many maximal ideals: you can take $\mathfrak{m}_n = (x - n, y^2 + n^2 + 1)$ for $n \in \mathbf{N}$.

Example 0.10. Suppose that we consider the ring

$$A = k[x, y, z, w]/(xy, xz)$$

Then we have two irreducible components corresponding to the prime ideals (x) and (y, z). The dimension of these irreducible components is 3 and 2. Thus the dimension of A is 3.