NOTES ON HEIGHTS

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1. Introduction

The way I am explaining and introducing the material is a little bit nonstandard although it covers all the usual ingredients. Just keep that in mind.

We will work consistently with what are called *logarithmic heights* in the literature.

2. Goal of the discussion

Let K be a field. We assume given for each $n \geq 0$ a function

$$h_n : \mathbf{P}^n(K) \longrightarrow \mathbf{R}$$

These functions have to satisfy some axioms (yet to be clarified).

Suppose given some projective variety X. Throughout variety will mean variety over K. In particular $\mathbf{P}^n$ will mean projective n-space over K. Suppose given a morphism of varieties

$$\varphi : X \longrightarrow \mathbf{P}^n$$

Then we get a function

$$h_\varphi = h_n \circ \varphi : X(K) \longrightarrow \mathbf{R}_{\geq 0}$$

and we get an invertible module $\mathcal{L}_\varphi = \varphi^*\mathcal{O}_{\mathbf{P}^n}(1)$. Our goal is to have enough axioms such that the following are true

(1) Given two morphisms $\varphi : X \to \mathbf{P}^n$ and $\psi : X \to \mathbf{P}^m$ such that $\mathcal{L}_\varphi \cong \mathcal{L}_\psi$ as invertible modules on X then there exists a constant $C > 0$ such that
$$|h_\varphi - h_\psi| \leq C$$
(2) given three morphisms $\varphi : X \to \mathbf{P}^n$, $\varphi' : X \to \mathbf{P}^{n'}$, and $\psi : X \to \mathbf{P}^m$ such that $\mathcal{L}_\varphi \otimes \mathcal{L}_{\varphi'} \cong \mathcal{L}_\psi$ as invertible modules on X then there exists a constant $C > 0$ such that
$$|h_\varphi + h_{\varphi'} - h_\psi| \leq C$$

Note that actually property (1) follows from property (2) as we can pick $\varphi' : X \to \mathbf{P}^0$ the constant morphism.

Of course, conditions (1) and (2) are trivially true if we pick each h_n bounded. But as soon as some h_n is not bounded, then it is not clear that (1) and (2) are satisfied.

3. Heights over the rationals

It turns out that over the rationals, i.e., when $K = \mathbf{Q}$ we can construct interesting natural functions
$$h_n : \mathbf{P}^n(\mathbf{Q}) \longrightarrow \mathbf{R}_{\geq 0}$$
Namely, suppose given a point
$$x = (x_0 : \ldots : x_n) \in \mathbf{P}^n(\mathbf{Q})$$
Then elementary number theory tells us there is a unique $c = c_x \in \mathbf{Q}^*_{>0}$ such that $cx_0, \ldots, cx_n \in \mathbf{Z}$ and $\gcd(cx_0, \ldots, cx_n) = 1$. We set
$$h_n(x) = \log(\max_{i=0,\ldots,n} |cx_i|), \quad c = c_x$$
For example we have
$$h_2([1/2 : 1/3 : 1/5]) = h_2([15 : 10 : 6]) = \log(15)$$

Lemma 3.1. *Let $x = (x_0 : \ldots : x_n)$ be a point of $\mathbf{P}^n(\mathbf{Q})$ and assume that $x_i \in \mathbf{Z}$ for all i. Then $h_n(x) \leq \max |x_i|$.*

Proof. This is true because $h_n(x)$ is exactly equal to $\max |x_i|$ divided by the gcd of the integers x_i. □

4. Heights over number fields

Very briefly, suppose we have a number field K. Then we want to define
$$h_n : \mathbf{P}^n(K) \longrightarrow \mathbf{R}_{\geq 0}$$
in exactly the same manner as in Section 3. This doesn't work for the following two reasons:

(1) since the ring of integers $\mathcal{O}_K$ isn't a PID we cannot represent every point $x \in \mathbf{P}^n(K)$ by a vector $(x_0, \ldots, x_n)$ with $x_i \in \mathcal{O}_K$ and $\gcd(x_0, \ldots, x_n) = 1$, and
(2) even if we could do this, then what are we going to use for $|x_i|$? Namely, there are multiple archimedean places and we don't want to just pick one!

The standard solution is to choose absolute values $||\cdot||_v$ for every place v of K satisfying the product formula $\prod_v ||c||_v = 1$ for all $c \in K^*$, and then to set

$$h_n(x) = \log\left(\prod\nolimits_v \max_{i=0,\ldots,n} ||x_i||_v\right) = \sum\nolimits_v \log\left(\max_{i=0,\ldots,n} ||x_i||_v\right)$$

whenever $x = (x_0 : \ldots : x_n) \in \mathbf{P}^n(K)$. See the discussion in Section 15 for a more general case.

Exercise 4.1. If $K = \mathbf{Q}$ and for the p-adic place v we take $||p||_v = p^{-1}$ and for the archimedian place v of $\mathbf{Q}$ we use the usual absolute value on $\mathbf{R}$, then the formula above recovers the formula given in Section 3.

Back to our general number field K. We can normalize the choice of the absolute values $||\cdot||_v$ such that these functions h_n agree with the ones in Section 3 via the inclusion $\mathbf{Q} \subset K$, then we find actually height functions

$$h_n : \mathbf{P}^n(\overline{\mathbf{Q}}) \longrightarrow \mathbf{R}_{\geq 0}$$

See discussion in lecture and see discussion in Section 19.

5. Heights of rational functions

Let k be a field (for example a finite field). Then we can set $K = k(t)$ the function field in 1-variable t over k. Exactly as in the case of $\mathbf{Q}$ we can define

$$h_n : \mathbf{P}^n(K) \longrightarrow \mathbf{R}_{\geq 0}$$

by a procedure involving clearing denominators. Namely, suppose given a point

$$x = (x_0 : \ldots : x_n) \in \mathbf{P}^n(K)$$

Then since $k[t]$ is a PID there is a scalar $c = c_x \in K^*$ unique up to k^* such that $cx_0, \ldots, cx_n \in k[t]$ and $\gcd(cx_0, \ldots, cx_n) = 1$. We set

$$h_n(x) = \max_{i=0,\ldots,n} \deg(cx_i), \quad c = c_x$$

The result is independent of the choice of c. For example we have

$$h_2([1/t : 1/(t-1) : 1]) = h_2([t-1 : t : t^2 - t]) = 2$$

6. Heights on function fields on curves

Let C be a nonsingular projective curve over a field k. Set $K = k(C)$. Then we have an interesting geometric way to define height functions

$$h_n : \mathbf{P}^n(K) \longrightarrow \mathbf{R}$$

as follows. A point $x \in \mathbf{P}^n(K)$ is a sequence $x = (f_0, \ldots, f_n)$ of elements of the function field K of C. Denote $V = \langle f_0, \ldots, f_n \rangle$ the k-subvector space of K generated by $f_0, \ldots, f_n$. If $\dim_k V = 1$, then we set $h_n(x) = 0$. If $\dim_k V > 1$, then in the lectures we constructed an invertible module $\mathcal{L} \subset \underline{K}$ generated by $f_0, \ldots, f_n$. Then we set

$$h_n(x) = \deg_C(\mathcal{L})$$

This has the following pleasing geometric interpretation: the point x corresponds to a unique morphism

$$x : C \longrightarrow \mathbf{P}^n_k$$

of varieties over k. Then $h_n(x) = \deg_C(x^*\mathcal{O}(1))$.

Exercise 6.1. If $K = k(t)$ and $C = \mathbf{P}^1_k$, then the construction above recovers the construction in Section 5.

Many of the properties of the height functions h_n for function fields of curves can easily be deduced from this geometric definition and a little bit of geometry.

7. Axiom 1 and the Segre map

Consider the variety $X = \mathbf{P}^n \times \mathbf{P}^{n'}$ together with the Segre embedding

$$\psi : X \longrightarrow \mathbf{P}^{nn'+n+n'}$$

which on coordinates is given by the rule

$$(x, y) = ((x_0 : \ldots : x_n), (y_0 : \ldots : y_{n'})) \longmapsto x \otimes y = (x_0y_0 : \ldots : x_iy_j : \ldots : x_ny_{n'})$$

Furthermore, denote $\varphi : X \to \mathbf{P}^n$ and $\varphi' : X \to \mathbf{P}^{n'}$ the projection morphisms.

Lemma 7.1. *With notation as above we have $\mathcal{L}_\varphi \otimes \mathcal{L}_{\varphi'} \cong \mathcal{L}_\psi$ as invertible modules on $X = \mathbf{P}^n \times \mathbf{P}^{n'}$.*

Proof. Discussed in the lecture. □

Thus if we want our goal to be true, more specifically if we want requirement (2) to hold in the case described above, we need to asssume

(Axiom 1) For every pair of integers $n, n' \geq 0$ there exists a constant $C = C(n, n') > 0$ such that for any points $x \in \mathbf{P}^n(K)$ and $y \in \mathbf{P}^{n'}(K)$ we have

$$|h_n(x) + h_{n'}(y) - h_{nn'+n+n'}(x \otimes y)| \leq C$$

where $x \otimes y$ is the point described above.

8. Axiom 1 for the rationals

This is true because suppose that we have $(x_0, \ldots, x_n) \in \mathbf{Z}^{n+1}$ with $\gcd(x_0, \ldots, x_n) = 1$ and that we have $(y_0, \ldots, y_{n'}) \in \mathbf{Z}^{n'+1}$ with $\gcd(y_0, \ldots, y_n) = 1$. Then of course we have that $x_iy_j \in \mathbf{Z}$ and $\gcd(x_iy_j) = 1$. Thus we see that

$$\begin{aligned} h_{nn'+n+n'}(x \otimes y) &= \log \max |x_iy_j| \\ &= \log(\max |x_i|)(\max |y_j|)) \\ &= \log(\max |x_i|) + \log(\max |y_j|) \\ &= h_n(x) + h_{n'}(y) \end{aligned}$$

and we have equality on the nose!

9. Axiom 1 for the rational functions

Here $K = k(t)$ and we can argue in exactly the same manner as in the case of the rational numbers.

10. Axiom 2 and change of coordinates

Consider the variety $X = \mathbf{P}^n$. Consider a **linear** morphism

$$\psi : X \longrightarrow \mathbf{P}^m$$

given by a matrix $A = (a_{ij})$ on coordinates as follows

$$x = (x_0 : \ldots : x_n) \longmapsto Ax = (\sum a_{0i}x_i : \sum a_{1i}x_i : \ldots : \sum a_{mi}x_i)$$

For this to make sense we need to make sure that there is no point x such that the output has all vanishing cooridates. In other words, we need to assume that the rank of the $(m+1) \times (n+1)$-matrix A has rank $n+1$. (In particular, this implies that $m \geq n$ of course.)

Lemma 10.1. *With notation as above we have $\mathcal{O}_{\mathbf{P}^n}(1) \cong \mathcal{L}_\psi$ as invertible modules on $X = \mathbf{P}^n$.*

Proof. Discussed in the lecture. □

Thus if we want our goal to be true, more specifically if we want requirement (1) to hold for $X = \mathbf{P}^n$, for $\varphi = \mathrm{id}_X$, for $\psi : X \to \mathbf{P}^m$ a linear morphism, then we need to asssume

(Axiom 2) For every pair of integers $m \geq n \geq 0$ and for $A \in \mathrm{Mat}((m+1)\times(n+1), K)$ of maximal rank, there exists a constant $C = C(A, n, m) > 0$ such that for any point $x \in \mathbf{P}^n(K)$ we have

$$|h_n(x) - h_m(Ax)| \leq C$$

where Ax is the point described above.

It turns out that we can reduce this a bit further.

Lemma 10.2. *In the situation above Axiom 2 holds if and only if the following conditions are satisfied*

(1) *For every pair of integers $m \geq n \geq 0$ there exists a constant $C = C(n, m) > 0$ such that for all $x \in \mathbf{P}^n(K)$ we have*

$$|h_n(x_0 : \ldots : x_n) - h_m(x_0 : \ldots : x_n : 0 : \ldots : 0)| \leq C$$

(2) *For every $n \geq i \geq 0$ and $c \in K^*$ there exists a constant $C = C(n, i, c)$ such that for all $x \in \mathbf{P}^n(K)$ we have*

$$|h_n(x_0 : \ldots : x_n) - h_n(x_0 : \ldots : cx_i : \ldots : x_n)| \leq C$$

(3) *For every $n > 0$, $i, j \in \{0, \ldots, n\}$, $i \neq j$ and $\lambda \in K^*$ there exists a constant $C = C(n, i, j, \lambda)$ such that for all $x \in \mathbf{P}^n(K)$ we have*

$$|h_n(x_0 : \ldots : x_n) - h_n(x_0 : \ldots : x_i + \lambda x_j : \ldots : x_n)| \leq C$$

Proof. See lectures. Hint: any square invertible matrix can be written as a product of elementary matrices. □

11. Axiom 2 for the rationals

Consider a pair of integers $m \geq n \geq 0$ and a matrix $A \in \mathrm{Mat}((m+1) \times (n+1), \mathbf{Q})$ of maximal rank. Let $d \geq 1$ be a common denominator for the coefficients of A, so dA has integer entries. Then we see that for $x \in \mathbf{P}^n(\mathbf{Q})$ the vectors

$$Ax \quad \text{and} \quad (dA)x$$

define the same point in $\mathbf{P}^m(\mathbf{Q})$. Thus we may assume A has coefficients in $\mathbf{Z}$.

Assume A has coefficients a_{ij} in $\mathbf{Z}$. Choose $C \geq 0$ such that $|a_{ij}| \leq C$ for all i, j. Suppose that we have $x = (x_0, \ldots, x_n) \in \mathbf{Z}^{n+1}$ with $\gcd(x_0, \ldots, x_n) = 1$. Then we see that

$$\begin{aligned} h_m(Ax) &\leq \log(\max\nolimits_{j=0,\ldots,m} |\sum\nolimits_i a_{ji}x_i|) \\ &\leq \log((n+1)C \max\nolimits_{i=0,\ldots,n} |x_i|) \\ &= \log((n+1)C) + h_n(x) \end{aligned}$$

Here the first inequality is Lemma 3.1. This gives us one of the two inequalities.

For the other inequality, since the integer matrix A has rank $m+1$, it follows from linear algebra that there exists a $(n+1) \times (m+1)$-matrix $B = (b_{ij})$ with coefficients in $\mathbf{Q}$ such that $BA = \mathbf{1}_{n+1}$. Choose an integer $e \geq 1$ such that $eb_{ij} \in \mathbf{Z}$. Choose a $C' > 0$ such that $|eb_{ij}| \leq C'$ for all i, j. With $x = (x_0, \ldots, x_n) \in \mathbf{Z}^{n+1}$ as above, writing $y = Ax$ so $y_j = \sum_i a_{ji}x_i$ are in $\mathbf{Z}$, we get

$$\begin{aligned} h_n(x) &= h_n(eBAx) \\ &= h_n(eBy) \\ &\leq \log(\max\nolimits_{k=0,\ldots,n} |\sum\nolimits_j eb_{kj}y_j|) \\ &\leq \log((m+1)C' \max\nolimits_{i=0,\ldots,n} |y_j|) \\ &= \log((m+1)C') + h_m(y) \\ &= \log((m+1)C') + h_m(Ax) \end{aligned}$$

Here the first inequality (on the third line) is Lemma 3.1. This proves the other inequality and the proof of Axiom 2 is done.

12. Axiom 2 for the rational functions

Here $K = k(t)$ and we can argue in exactly the same manner as in the case of the rational numbers.

13. Useful facts about invertible modules

If X is a projective variety over K we say that an invertible $\mathcal{O}_X$-module $\mathcal{L}$ is *very ample* if there exist sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ which generate $\mathcal{L}$ such that the morphism

$$\varphi_{\mathcal{L}, s_0, \ldots, s_n} : X \longrightarrow \mathbf{P}^n$$

(discussed in the lectures) is a closed immersion.

Fact I. If $\mathcal{L}$ is very ample and if $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ span the K-vector space $\Gamma(X, \mathcal{L})$, then $s_0, \ldots, s_n$ generate $\mathcal{L}$ and $\varphi_{\mathcal{L}, s_0, \ldots, s_n}$ is a closed immersion.

Fact II. For every invertible module $\mathcal{L}$ there exist very ample invertible modules $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{L} \cong \mathcal{M} \otimes \mathcal{N}^{\otimes -1}$.

Fact III. If $\mathcal{L}$ is globally generated and $\mathcal{N}$ is very ample, then $\mathcal{L} \otimes \mathcal{N}$ is very ample.

Fact IV. For every invertible module $\mathcal{L}$ and global sections $s_0, \ldots, s_n$ which generate $\mathcal{L}$ there exists a very ample invertible module $\mathcal{N}$ and global sections $t_0, \ldots, t_m \in \Gamma(X, \mathcal{N})$ which span the K-vector space $\Gamma(X, \mathcal{N})$ and such that $s_i t_j$ span the K-vector space $\Gamma(X, \mathcal{L} \otimes \mathcal{N})$.

14. Heights associated to line bundles

In this section we will show: given a field K and a collection of functions h_n as in Section 2 satisfying Axioms 1 and 2 then for every projective variety X and invertible $\mathcal{O}_X$-module $\mathcal{L}$ we obtain function $h_{\mathcal{L}} : X(K) \to \mathbf{R}$ well defined up to a constant[1]. This construction will satisfy

1. $h_{\mathcal{O}(1)} = h_n$ on $X = \mathbf{P}^n$
2. more generally if $\mathcal{L} = \varphi^* \mathcal{O}(1)$ then we have $h_{\mathcal{L}} = h_{\varphi}$ as defined above,
3. for X and a pair of invertible modules $\mathcal{L}$ and $\mathcal{L}'$ we have the equality of functions $h_{\mathcal{L}} + h_{\mathcal{L}'} = h_{\mathcal{L} \otimes \mathcal{L}'}$,
4. given a morphism $f : Y \to X$ of projective varieties we have $h_{f^*\mathcal{L}} = h_{\mathcal{L}} \circ f$.

In the rest of this section we explain the construction.

Step 1. Let $\mathcal{L}$ be very ample on X. Then we claim that there exists a well defined function $h_{\mathcal{L}} : X \to \mathbf{R}$ constructed as follows: pick any $n \geq 0$ and global sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ which span $\Gamma(X, \mathcal{L})$ as a K-vector space. By Fact I the morphism $\varphi_{\mathcal{L}, s_0, \ldots, s_n}$ is defined and we may set $h_{\mathcal{L}} = h_n \circ \varphi_{\mathcal{L}, s_0, \ldots, s_n}$. Why is this well defined? To see this it suffices to show that if $t_0, \ldots, t_m \in \Gamma(X, \mathcal{L})$ is a basis then $h_n \circ \varphi_{\mathcal{L}, s_0, \ldots, s_n} - h_m \circ \varphi_{\mathcal{L}, t_0, \ldots, t_m}$ is bounded. Namely, we can write $s_i = \sum a_{ij} t_j$. Since $t_0, \ldots, t_m$ is a basis of and since s_i span the K-vector space $\Gamma(X, \mathcal{L})$ we see that A has maximal rank! We have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\varphi_{\mathcal{L}, t_0, \ldots, t_m}"'] \arrow[dr, "\varphi_{\mathcal{L}, s_0, \ldots, s_n}"'] & \mathbf{P}^m \arrow[d, "x \mapsto Ax"] \\ & \mathbf{P}^n \end{tikzcd}$$

Thus the bound we want follows from Axiom 2.

Step 2. Next, suppose that

1. $\mathcal{L}$ is a invertible module,
2. $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ are global sections which generate $\mathcal{L}$, and
3. $s_{n+1}, \ldots, s_{n+n'} \in \Gamma(X, \mathcal{L})$ are some additional global sections.

By Fact III and IV we can find a very ample invertible module $\mathcal{N}$ such that $\mathcal{L} \otimes \mathcal{N}$ is very ample and global sections $t_0, \ldots, t_m \in \Gamma(X, \mathcal{N})$ which span the K-vector space $\Gamma(X, \mathcal{N})$ and such that $s_i t_j$ span the K-vector space $\Gamma(X, \mathcal{L} \otimes \mathcal{N})$. Then we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "(\varphi_{\mathcal{L}, s_i}, \varphi_{\mathcal{N}, t_j})"] \arrow[rr, bend right, "\varphi_{\mathcal{L} \otimes \mathcal{N}, s_i t_j}"'] & \mathbf{P}^n \times \mathbf{P}^m \arrow[r] & \mathbf{P}^{nm+n+m} \end{tikzcd}$$

[1] Technically, the symbol $h_{\mathcal{L}}$ is an element of the quotient $\mathrm{Maps}(X(K), \mathbf{R})/(\text{bounded maps})$.

Thus by Axiom 1 we see that

$$h_n \circ \varphi_{\mathcal{L},s_0,\ldots,s_n} = h_{\mathcal{L}\otimes\mathcal{N}} - h_{\mathcal{N}}$$

up to bounded functions where the functions $h_{\mathcal{L}\otimes\mathcal{N}}$ and $h_{\mathcal{N}}$ are as defined in Step 1. But note that the same choice of $\mathcal{N}, t_0, \ldots, t_m$ works for $\mathcal{L}, s_0, \ldots, s_{n+n'}$. Hence we also conclude that

$$h_n \circ \varphi_{\mathcal{L},s_0,\ldots,s_n} - h_{n+n'} \circ \varphi_{\mathcal{L},s_0,\ldots,s_{n+n'}}$$

is bounded as both are equal to $h_{\mathcal{L}\otimes\mathcal{N}} - h_{\mathcal{N}}$ up to bounded functions.

Step 3. Let $\mathcal{L}$ be a very ample invertible module and let $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ are global sections which generate $\mathcal{L}$. Since we can find additional global sections $s_{n+1}, \ldots, s_{n+n'} \in \Gamma(X, \mathcal{L})$ such that $s_0, \ldots, s_{n+n'}$ generate the K-vector space $\Gamma(X, \mathcal{L})$ we conclude from Step 2 that

$$h_{\mathcal{L}} - h_n \circ \varphi_{\mathcal{L},s_0,\ldots,s_n}$$

is bounded where $h_{\mathcal{L}}$ is as in Step 1.

Step 4. Let $\mathcal{L} = \mathcal{N} \otimes \mathcal{M}$ with all three invertible modules very ample. Choose a basis $s_0, \ldots, s_n$ of $\Gamma(X, \mathcal{N})$. Choose a basis $t_0, \ldots, t_m$ of $\Gamma(X, \mathcal{M})$. Then we do not know if $s_i t_j$ form a basis of $\Gamma(X, \mathcal{N} \otimes \mathcal{M})$ or even if they generate $\Gamma(X, \mathcal{N} \otimes \mathcal{M})$ as a K-vector space. But we do know that they generate the invertible module $\mathcal{N} \otimes \mathcal{M}$. Hence by Step 3 we conclude that

$$h_{\mathcal{L}} = h_{nm+n+m} \circ \varphi_{\mathcal{N}\otimes\mathcal{M},s_i t_j}$$

up to bounded functions. On the other hand we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "(\varphi_{\mathcal{N},s_i},\varphi_{\mathcal{M},t_j})"] \arrow[rr, bend right, "\varphi_{\mathcal{N}\otimes\mathcal{M},s_i t_j}"'] & \mathbf{P}^n \times \mathbf{P}^m \arrow[r] & \mathbf{P}^{nm+n+m} \end{tikzcd}$$

Thus by Axiom 1 we see that

$$h_{\mathcal{L}} = h_{nm+n+m} \circ \varphi_{\mathcal{N}\otimes\mathcal{M},s_i t_j} = h_{\mathcal{N}} + h_{\mathcal{M}}$$

up to bounded functions.

Step 5. Next, suppose that $\mathcal{L}$ is any invertible module. By Fact II there exist very ample invertible modules $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{L} \cong \mathcal{M} \otimes \mathcal{N}^{\otimes -1}$. Then we set $h_{\mathcal{L}} = h_{\mathcal{M}} - h_{\mathcal{N}}$ where $h_{\mathcal{M}}$ and $h_{\mathcal{N}}$ are as above. To show that this is well defined, suppose that $\mathcal{L} \cong \mathcal{K} \otimes \mathcal{J}^{\otimes -1}$. for another pair of very ample invertible modules $\mathcal{K}$ and $\mathcal{J}$. Then we get

$$\mathcal{K} \otimes \mathcal{N} \cong \mathcal{M} \otimes \mathcal{J}$$

We know that these are very ample invertible modules by Fact III. Hence by the previous paragraph we conclude that

$$\varphi_{\mathcal{K}} + \varphi_{\mathcal{N}} = \varphi_{\mathcal{M}} + \varphi_{\mathcal{J}}$$

up to a bounded function and we conclude that our prescription is well defined.

Step 6. If $\mathcal{L}$ and $\mathcal{L}'$ are invertible modules and we write $\mathcal{L} \cong \mathcal{M} \otimes \mathcal{N}^{\otimes -1}$ and $\mathcal{L}' \cong \mathcal{K} \otimes \mathcal{J}^{\otimes -1}$ then we see that

$$\mathcal{L} \otimes \mathcal{L}' = (\mathcal{M} \otimes \mathcal{K}) \otimes (\mathcal{N} \otimes \mathcal{J})^{\otimes -1}$$

By Fact II the invertible modules $\mathcal{M} \otimes \mathcal{K}$ and $\mathcal{N} \otimes \mathcal{J}$ are very ample. We find that by construction in Step 5 that

$$h_{\mathcal{L}\otimes\mathcal{L}'} = h_{\mathcal{M}\otimes\mathcal{K}} - h_{\mathcal{N}\otimes\mathcal{J}} = h_{\mathcal{M}} + h_{\mathcal{K}} - h_{\mathcal{N}} - h_{\mathcal{J}} = h_{\mathcal{L}} + h_{\mathcal{L}'}$$

as desired.

Step 7. We still have to prove the functoriality in (4). If $\mathcal{L}$ is globally generated, say by $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ then we see that

$$\varphi_{\mathcal{L}, s_0, \ldots, s_n} \circ f = \varphi_{f^*\mathcal{L}, f^*s_0, \ldots, f^*s_n}$$

and hence by Step 3 used twice we obtain that $h_{\mathcal{L}} \circ f = h_{f^*\mathcal{L}}$. Since in general we can write $\mathcal{L}$ as a difference of globally generated (very ample) invertible modules we conclude from the additivity of height functions shown in Step 6 that $h_{\mathcal{L}} \circ f = h_{f^*\mathcal{L}}$ holds for an arbitrary invertible module on X.

15. Heights from local functions

In Brian Conrad's paper "Chow's K/k-trace..." and in Moriwaki's paper "Arithmetic height functions..." there is a clever construction[2] of heights by analogy with the number field case.

Let K be a field. Suppose that we have a set $M = M_K$ and for each $v \in M$ a function

$$|| \cdot ||_v : K \longrightarrow \mathbf{R}_{\geq 0}$$

We assume

(1) for $c \in K$ we have $||c||_v = 0 \Leftrightarrow c = 0$,
(2) for $c, c' \in K$ we have $||cc'||_v = ||c||_v ||c'||_v$,
(3) for $c \in K^*$ we have $||c||_v = 1$ for all but a finite number of $v \in M_K$ and we have the product formula $\prod_{v \in M} ||c||_v = 1$.

Then the formula

$$h_n(x) = \log\left(\prod\nolimits_v \max\nolimits_{i=0,\ldots,n} ||x_i||_v\right) = \sum\nolimits_v \log\left(\max\nolimits_{i=0,\ldots,n} ||x_i||_v\right)$$

whenever $x = (x_0 : \ldots : x_n) \in \mathbf{P}^n(K)$ is at least well defined. Axiom 1 is automatic with this choice. If Axiom 2 holds just for the morphism

$$\mathbf{P}^1 \longrightarrow \mathbf{P}^1, \quad (1 : a) \longmapsto (1 : 1 + a)$$

then there exists a constant $C > 0$ such that

$$\sum\nolimits_v \log\left(||1 + a||_v\right) \leq \sum\nolimits_v \log(\max\{1, ||a||_v\}) + C \tag{15.0.1}$$

for all $a \in K^*$. Thus it appears that we need some way to compare $||1 + a||_v$ in terms of the maximum of 1 and $||a||_v$. For example, suppose we have some real numbers $\epsilon_v \geq 1$ such that

$$||1 + a||_v \leq \epsilon_v \max\{1, ||a||_v\} \tag{15.0.2}$$

for all $a \in K^*$ and such that we have

$$C = \sum\nolimits_{v \in M} \log(\epsilon_v) < \infty$$

[2]It seems Moriwaki's height function actually comes from a slightly more general construction where the local functions are allowed to take negative values and perhaps do not exactly statisfy an inequality as in (15.0.2).

Then we obtain (15.0.1). For example, if $||\cdot||_v$ is an absolute value, then we have the triangle inequality $||a+b||_v \leq ||a||_v + ||b||_v$ and we can take $\epsilon_v = 2$. If $||\cdot||_v$ is a non-Archimedian absolute value, then we have $||a+b||_v \leq \max(||a||_v, ||b||_v)$ and we can take $\epsilon_v = 1$. Thus if all the $||\cdot||_v$ are absolute values, then we just need to assume we have only finitely many $v \in M$ where $||\cdot||_v$ is archmedian (since otherwise the condition that $C < \infty$ is violated); of course this is exactly what happens in the number field case.

Lemma 15.1. *Axiom 2 holds if we have constants ϵ_v as above.*

Proof. We use the criterion of Lemma 10.2. For the linear embeddings $\mathbf{P}^n \to \mathbf{P}^m$ sending $x = (x_0 : \ldots : x_n)$ to $(x_0 : \ldots : x_n : 0 : \ldots : 0)$ we actually have equality with the definition of h_n as above.. The existence of a bound is easy for diagonal matrices (see lectures for details). Finally, the bound for a morphism of the form

$$(x_0 : \ldots : x_n) \longmapsto (x_0 : \ldots : x_{i-1} : x_i + \lambda x_j : x_{i+1} : \ldots : x_n)$$

where $i \neq j$ is done by a direct computation using the constants ϵ_v (see lecture for details). □

Remark 15.2. Let K be a field. Let $||\cdot|| : K \to \mathbf{R}_{\geq 0}$ be a map such that (1) for $c \in K$ we have $||c|| = 0 \Leftrightarrow c = 0$, (2) for $c, c' \in K$ we have $||cc'|| = ||c|| \cdot ||c'||$, and (3) there exists a real number $\epsilon \geq 1$ such that $||1 + a|| \leq \epsilon \max\{1, ||a||\}$ for all $a \in K^*$. I don't have an example of this where $||\cdot||$ isn't a power of an absolute value. Do you?

16. Positivity

In the rational number field and rational function field cases the functions we have constructed have nonnegative values. However, in those cases it turns out we have the following even stronger fact.

Lemma 16.1. *Let $K = \mathbf{Q}$ or $K = k(t)$ with h_n as constructed in Section 3 or 5. Given X and $\mathcal{L}$ denote $B \subset X$ the base locus of of $\mathcal{L}$. Then $h_{\mathcal{L}}$ restricted to $X(K) \setminus B(K)$ is bounded from below*[3].

Proof. See discussion in lecture □

17. Points of bounded height

If $K = \mathbf{Q}$ or $K = k(t)$ with k finite(!) the sets

$$\{x \in \mathbf{P}^n(K) : h_n(x) \leq C\}$$

are finite for any real number C. See discussion in lecture.

18. Heights and abelian varieties

Let K be a field and let $\{h_n\}$ be a collection of functions satisfying axioms 1 and 2. Let X be an abelian variety over K. Recall that this means X is a smooth projective variety over K, we are given a K-rational point $O \in X(K)$, we are given a morphism $m : X \times X \to X$ of varieties over K, and we are given an isomorphism $i : X \to X$ such that

(1) $m(x, m(y, z)) = m(m(x, y), z)$ for all scheme valued points x, y, z of X,

[3] Since our functions are only well defined up to a constant, this is the best we can ask for.

(2) $m(x, O) = m(O, x) = x$ for all scheme valued points x of X,
(3) $m(i(x), x) = m(x, i(x)) = O$ for all scheme valued points x of X.

Under these assumptions it is always the case that $m(x, y) = m(y, x)$. Thus we write $m(x, y) = x + y$ and $i(x) = -x$.

Fact. For any invertible module $\mathcal{L}$ on X we have

$$\mathcal{O}_{X\times X\times X} \cong m_{123}^*\mathcal{L} \otimes m_{12}^*\mathcal{L}^{\otimes -1} \otimes m_{13}^*\mathcal{L}^{\otimes -1} \otimes m_{23}^*\mathcal{L}^{\otimes -1} \otimes p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \otimes p_3^*\mathcal{L}$$

where the morphisms $m_{123}, m_{12}, \ldots : X \times X \times X \to X$ are the following ones

$$m_{123}(x, y, z) = x + y + z, \quad m_{12}(x, y, z) = x + y$$

similarly for m_{13} and m_{23} and p_1, p_2, and p_3 are the projections.

Conclusion: from our height machine in Section 14 we see immediately that there exists a constant $C = C(X, \mathcal{L})$ such that

$$|h_\mathcal{L}(x+y+z) - h_\mathcal{L}(x+y) - h_\mathcal{L}(x+z) - h_\mathcal{L}(y+z) + h_\mathcal{L}(x) + h_\mathcal{L}(y) + h_\mathcal{L}(z)| < C$$

for all $x, y, z \in X(K)$. This isn't quite enough to show that $h_\mathcal{L}$ is a quadratic function up to a bounded function. However, suppose that $i^*\mathcal{L} \cong \mathcal{L}$; we will say $\mathcal{L}$ is *symmetric*. Then we also have a constant C' such that

$$|h_\mathcal{L}(x) - h_\mathcal{L}(-x)| < C'$$

for all $x \in X(K)$ since $-x = i(x)$ by our conventions above.

Writing out what the long equality above means for $x, y, z = x, x, -x$ and using the bound from symmetry of $\mathcal{L}$ in the previous paragraph we obtain

$$|h_\mathcal{L}(2x) - 4h_\mathcal{L}(x)| < C + C'$$

for all $x \in X(K)$. This suggests considering for $n \geq 1$ the function

$$g_n : X(K) \to \mathbf{R}, \quad g_n(x) = \frac{h_\mathcal{L}(2^n x)}{4^n}$$

Then one shows

(1) $|g_n(O)| \leq |h_\mathcal{L}(O)|/4^n$,
(2) $|h_\mathcal{L}(x) - g_n(x)| \leq C + C'$ for all n for all $x \in X(K)$,
(3) we have
$$|g_n(x+y+z) - g_n(x+y) - g_n(x+z) - g_n(y+z) + g_n(x) + g_n(y) + g_n(z)| < C/4^n$$
for all $x, y, z \in X(K)$,
(4) we have $|g_n(x) - g_n(-x)| < C'/4^n$ for all $x \in X(K)$, and
(5) $|g_n(2x) - 4g_n(x)| < (C + C')/4^n$ for all $x \in X(K)$.

Then for any finitely generated subgroup $A \subset X(K)$ we can pick a sequence $n_1 < n_2 < n_3 < \ldots$ such that $g_{n_i}|_A$ converges to a function g which is quadratic, i.e., satisfies $g(O) = 0$, $g(-x) = g(x)$, and the map $\langle , \rangle : A^2 \to \mathbf{R}$ given by $\langle x, y\rangle = g(x+y) - g(x) - g(y)$ is symmetric bilinear[4]. Conversely, we then have $g(x) = \frac{1}{2}\langle x, x\rangle$. Now note that the bilinear from $\langle , \rangle$ on A is the unique bilinear map such that

$$A \to \mathbf{R}, \quad x \longmapsto h_\mathcal{L}(x) - \frac{1}{2}\langle x, x\rangle = h_\mathcal{L}(x) - g(x)$$

[4]Because we have $\langle x+y, z\rangle - \langle x, z\rangle - \langle y, z\rangle = g(x+y+z) - g(x+y) - g(x+z) - g(y+z) + g(x) + g(y) + g(z)$ which is zero on A^3 by our limit construction of g.

is bounded (the reason being that different quadratic functions have unbounded difference!). By the way, observe that the bound for $|h_{\mathcal{L}}(x) - g(x)|$ is $C + C'$ independent of our choice of finitely generated subgroup A. Since $X(K)$ is the union of its finitely generated subgroups, we conclude g and $\langle , \rangle$ are defined canonically on all of $X(K)$.

Proposition 18.1. *Let X, O, m, i be an abelian variety over K. Let $\mathcal{L}$ be a symmetric invertible module on X. There exists a uniquely defined symmetric bilinear form*

$$\langle , \rangle : X(K) \longrightarrow \mathbf{R}$$

and a constant $C > 0$ such that

$$|h_{\mathcal{L}}(x) - \frac{1}{2}\langle x, x\rangle| < C$$

for all $x \in X(K)$.

Proof. See discussion above. $\square$

Remark 18.2. Given any abelian variety X, O, m, i over a field, it possesses at least one very ample invertible module which is symmetric. Namely, let $\mathcal{N}$ be any very ample invertible module. Then $i^*\mathcal{N}$ is very ample too, because i is an automorphism of X. By Fact III of Section 13 we see that $\mathcal{L} = \mathcal{N} \otimes i^*\mathcal{N}$ is very ample. Of course $\mathcal{L}$ is symmetric by construction.

19. Appendix: induced height on the algebraic closure

Don't read this!

Suppose we have a field K and functions h_n satisfying Axioms 1 and 2. Let $\overline{K}$ be the algebraic closure of K. It seems that there is a canonical way to extend the functions h_n (modulo bounded functions) to functions

$$\overline{h}_n : \mathbf{P}^n(\overline{K}) \longrightarrow \mathbf{R}_{\geq 0}$$

satisfying Axioms 1 and 2 although we haven't found this in the literature. In this section we discuss how to start doing this.

For example, we can define a map

$$\overline{h}_1 : \mathbf{P}^1(\overline{K}) \longrightarrow \mathbf{R}_{\geq 0}$$

as follows. Suppose that $x = (x_0 : x_1) \in \mathbf{P}^1(\overline{K})$. Let d be the minimal degree of a nonzero homogeneous polynomial $F \in K[T_0, T_1]$ such that $F(x_0, x_1) = 0$. Note that F is unique up to a scalar. For example, if $x_0 \neq 0$, then d is the degree of the field extension of K generated by $\alpha = x_1/x_0 \in \overline{K}$ and $F(1, T)$ is a multiple of the minimal polynomial of α. We write

$$F = a_0 T_0^d + a_1 T_0^{d-1} T_1 + \ldots + a_d T_1^d$$

for some $a_i \in K$. Since F is well defined up to a scalar we see that the point $(a_0 : \ldots : a_d) \in \mathbf{P}^d(K)$ is well defined. Then we set

$$\overline{h}_1(x) = \frac{h_d(a_0 : \ldots : a_d)}{d}$$

Example 19.1. If $x \in \mathbf{P}^1(K)$ then $d = 1$ and we can take $F = x_1T_0 - x_0T_1$ and we see that

$$\overline{h}_1(x) = h_1(x_1 : -x_0)$$

which by Axiom 2 differs from $h_1(x)$ by a bounded amount.

Next, consider the surjective finite morphism

$$\pi : \mathbf{P}^1 \times \mathbf{P}^1 \longrightarrow \mathbf{P}^2, \quad (x, y) \longmapsto (x_0y_0 : x_0y_1 + x_1y_0 : x_1y_1)$$

By Axioms 1 and 2 we have that $h_1(x) + h_1(y)$ and $h_2(\pi(x, y))$ differ by a constant for $x \in \mathbf{P}^1(K)$ and $y \in \mathbf{P}^1(K)$. Motivated by this we define

$$\overline{h}_2 : \mathbf{P}^1(\overline{K}) \longrightarrow \mathbf{R}_{\geq 0}$$

as follows. Given $z \in \mathbf{P}^2(\overline{K})$ pick $x \in \mathbf{P}^1(\overline{K})$ and $y \in \mathbf{P}^1(\overline{K})$ with $\pi(x, y) = z$. Then we set

$$\overline{h}_2(z) = \overline{h}_1(x) + \overline{h}_1(y)$$

Now we have to check something: why is this well defined? Well, since π has degree 2 the only problem is if there is a second point mapping to z by π. But this can only happen if $x \neq y$ and then the second point is the point (y, x). Since the formula above is symmetric in x and y this fine.

Example 19.2. Another sanity check is the following: suppose that we look at all $z \in \mathbf{P}^2(K)$. Then we want to make sure, as in Example 19.1 that $\overline{h}_2(z)$ and $h_2(z)$ have bounded difference. To see this we deal with two cases

(1) (x, y) is in $\mathbf{P}^1(K) \times \mathbf{P}^1(K)$. In this case $h_2(z)$ differs from $h_1(x) + h_1(y)$ by a bounded amount, $h_1(x)$ differs from $\overline{h}_1(x)$ by a bounded amount, and $h_1(y)$ differs from $\overline{h}_1(y)$ by a bounded amount. Thus this case is fine.

(2) (x, y) is defined over a quadratic extension of K. In this case a calculation shows the equation F we use to compute $\overline{h}_2(x)$ is

$$z_2T_0^2 - z_1T_0T_1 + z_0T_1^2$$

We conclude that $\overline{h}_1(x) = (1/2)h_2(z_2 : -z_1 : z_0)$. The exact same result holds for y by symmetry. Hence we see that $\overline{h}_2(z) = h_2(z_2 : -z_1 : z_0)$ which differs from $h_2(z)$ by a bounded amount.