The Fourier transform

1 Structure of the group algebra

Before we begin, we make some general remarks about algebras. Let k be a field and let A be a k-vector space. We say (somewhat informally) that A is a *k-algebra* if there is a k-bilinear form $A \times A \to A$, whose value at (a, b) we denote by ab. Bilinearity implies the left and right distributive laws (for all $a, b, a_1, a_2, b_1, b_2 \in A$)

$$(a_1 + a_2)b = a_1 b + a_2 b;$$
$$a(b_1 + b_2) = ab_1 + ab_2,$$

as well as the property that, for all $a, b \in A$ and $t \in k$,

$$(ta)b = a(tb) = t(ab).$$

Usually we shall just call A an algebra if the field k is clear from the context. The algebra A is *associative* if multiplication is associative i.e. for all $a, b, c \in A$, $(ab)c = a(bc)$, and *unital* if there is a multiplicative identity, i.e. an element usually denoted by 1 such that, for all $a \in A$, $1a = a1 = a$. Note that, in this case, $1 = 0 \iff A = \{0\}$. Otherwise, the map $k \to A$ defined by $t \mapsto t{\cdot}1$ is injective and identifies k with the subset $k{\cdot}1 = \{t{\cdot}1 : t \in k\}$ of A. For us, all algebras will be associative and unital (although there are many interesting classes of non-associative algebras). A *k-algebra homomorphism* $f\colon A \to B$ is a function from A to B which is both a k-linear map and a ring homomorphism; equivalently, f is k-linear and $f(ab) = f(a)f(b)$ for all $a, b \in A$. If A and B are unital, then we will also require that $f(1) = 1$. The algebra homomorphism f is an *isomorphism* if it is a bijection. In this case, f^{-1} is also an algebra homomorphism. A *subalgebra* A' of A is defined in the obvious way, as a vector subspace closed under multiplication. If A is unital then we also require that $1 \in A'$. In this case, $k \cdot 1$ is a subalgebra of A and is in fact the smallest subalgebra of A.

Definition 1.1. The *center* ZA of A is the set of elements which commute with every element of A:

$$ZA = \{a \in A : ab = ba \text{ for all } b \in A\}.$$

It is easy to check from the definitions that ZA is a subalgebra of A. Note that, if A is unital, then $1 \in ZA$, and more generally the subalgebra $k \cdot 1$ is contained in ZA.

Example 1.2. 1) The set $\mathbb{M}_d(k)$ of $d \times d$ matrices with coefficients in k is an associative, unital k-algebra, with multiplicative identity the identity matrix I. It is a linear algebra fact that the center of $\mathbb{M}_d(k)$ is exactly $k \cdot I = \{tI : t \in k\}$. In other words, the only $d \times d$ matrices which commute with all $d \times d$ matrices are scalar multiples of the identity matrix.

2) The group algebra $k[G]$ is an associative, unital k-algebra, with multiplicative identity $1 = 1 \cdot 1$, where the first 1 is the multiplicative identity in k and the second is the multiplicative identity in G. We can identify $k[G]$ with $L^2(G)$ (for $k = \mathbb{C}$) and multiplication with convolution of functions. We shall describe the center of $k[G]$ shortly.

3) Given two algebras A_1 and A_2, we can define the *product algebra* $A_1 \times A_2$ to be the Cartesian product as a vector space together with componentwise multiplication, i.e. given by

$$(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2).$$

Perhaps confusingly, we write the product algebra as $A_1 \times A_2$ and not $A_1 \oplus A_2$, because the product algebra as we have defined it is a **product** in the category of k-algebras, not a **coproduct**. (The coproduct of A_1 and A_2 is $A_1 \otimes A_2$.) Concretely, what this means is that, if A is an algebra and $f_1 \colon A \to A_1$ and $f_2 \colon A \to A_2$ are algebra homomorphisms, then $(f_1, f_2) \colon A \to A_1 \times A_2$ is an algebra homomorphism, and every algebra homomorphism from A to $A_1 \times A_2$ arises in this way.

It is easy to see that $A_1 \times A_2$ is associative $\iff$ A_1 and A_2 are associative, and that $A_1 \times A_2$ is unital $\iff$ A_1 and A_2 are unital, in which case the multiplicative identity in $A_1 \times A_2$ is $(1, 1)$. Finally, the center $Z(A_1 \times A_2)$ is $ZA_1 \times ZA_2$.

Definition 1.3. Let A be an associative and unital k-algebra. A *representation* of A on a vector space V is a k-algebra homomorphism $\rho \colon A \to \operatorname{End} V = \operatorname{Hom}(V, V)$.

Note that, if $t \in k$, then $\rho(t\alpha)(v) = t(\rho(\alpha))(v) = t(\rho(\alpha)(v))$. In particular, since A is unital, $\rho(t \cdot 1)(v) = tv$ and so the representation is compatible with, and determines, the vector space structure on V in the obvious sense. Using this, it is straightforward to show that a representation of A on V is the same thing as a (left) A-module V.

Now suppose that G is a finite group and that ρ_V is a G-representation. We claim that there is a natural way to extend ρ_V to give a representation of the group algebra $\mathbb{C}[G]$ (also denoted ρ_V) as follows: define

$$\rho_V(\sum_{g\in G} t_g \cdot g) = \sum_{g\in G} t_g \rho_V(g) \in \operatorname{End} V.$$

Viewing $\mathbb{C}[G]$ as $L^2(G)$, this formula gives

$$\rho_V(f) = \sum_{g\in G} f(g)\rho_V(g) = F_{V,f},$$

in the notation of the handout "Characters II," p. 1.

Lemma 1.4. *With notation as above, ρ_V is an algebra homomorphism from $\mathbb{C}[G]$ to $\operatorname{End} V$.*

Proof. This is essentially just a consequence of the way multiplication is defined in $\mathbb{C}[G]$. We have

$$\begin{aligned}\rho_V\left(\sum_{g\in G} t_g \cdot g \sum_{g\in G} s_g \cdot g\right) &= \rho_V\left(\sum_{g\in G}\left(\sum_{h_1h_2=g} t_{h_1}s_{h_2}\right)\cdot g\right)\\ &= \sum_{g\in G}\left(\sum_{h_1h_2=g} t_{h_1}s_{h_2}\right)\rho_V(g).\end{aligned}$$

On the other hand,

$$\begin{aligned}\rho_V\left(\sum_{g\in G} t_g \cdot g\right)\rho_V\left(\sum_{g\in G} s_g \cdot g\right) &= \left(\sum_{g\in G} t_g\rho_V(g)\right)\left(\sum_{g\in G} s_g\rho_V(g)\right)\\ &= \sum_{h_1,h_2\in G} t_{h_1}s_{h_2}\rho_V(h_1)\rho_V(h_2) = \sum_{h_1,h_2\in G} t_{h_1}s_{h_2}\rho_V(h_1h_2).\end{aligned}$$

By grouping together all the terms $t_{h_1}s_{h_2}\rho_V(h_1h_2)$ in the last summation for which $h_1h_2 = g$, we have

$$\sum_{h_1,h_2\in G} t_{h_1}s_{h_2}\rho_V(h_1h_2) = \sum_{g\in G}\left(\sum_{h_1h_2=g} t_{h_1}s_{h_2}\right)\cdot\rho_V(g).$$

Comparing, we see that ρ_V is a homomorphism as desired. $\square$

Remark 1.5. 1) In fact, every algebra representation of $\mathbb{C}[G]$, i.e. every algebra homomorphism from $\mathbb{C}[G]$ to $\operatorname{End} V$ where V is a vector space, arises in this way: given an algebra homomorphism $\rho_V \colon \mathbb{C}[G] \to \operatorname{End} V$, we can restrict ρ_V to $G \subseteq \mathbb{C}[G]$. Then $\rho_V(g)$ is invertible, since $\rho_V(g)\rho_V(g^{-1}) = \rho_V(gg^{-1}) = \rho_V(1) = \mathrm{Id}$, and then clearly the restriction of ρ_V to G defines a homomorphism $G \to \operatorname{Aut} V$.
2) Viewing $\mathbb{C}[G]$ as $L^2(G)$, the lemma says that, for all $f_1, f_2 \in L^2(G)$,

$$\rho_V(f_1 * f_2) = \rho_V(f_1) \cdot \rho_V(f_2),$$

where the last product is composition in $\operatorname{End} V$ (or matrix multiplication after choosing a basis to identify $\operatorname{End} V$ with $\mathbb{M}_d(\mathbb{C})$).
3) If V and W are two representations, then we can define

$$(\rho_V, \rho_W) \colon \mathbb{C}[G] \to \operatorname{End} V \times \operatorname{End} W$$

as in Example 1.2(3). There is also a natural algebra homomorphism

$$\operatorname{End} V \times \operatorname{End} W \to \operatorname{End}(V \oplus W)$$

which sends a pair (F_1, F_2) to the linear map $F_1 \oplus F_2$ (compare the handout on linear algebra, comment after Remark 7.4). Clearly the composition

$$\mathbb{C}[G] \xrightarrow{(\rho_V, \rho_W)} \operatorname{End} V \times \operatorname{End} W \to \operatorname{End}(V \oplus W)$$

is $\rho_{V \oplus W}$.

Example 1.6. 1) For the trivial representation $\rho = \rho_{\mathbb{C}(1)}$, $\rho(g) = 1 \in \mathbb{C}$ for every $g \in G$. Hence $\rho(\sum_{g \in G} t_g \cdot g) = \sum_{g \in G} t_g$, and it is not hard to check directly that this defines a $\mathbb{C}$-algebra homomorphism from $\mathbb{C}[G]$ to $\mathbb{C}$.
2) Let $V = \mathbb{C}[G]$ be the regular representation, so that $\rho_V = \rho_{\mathrm{reg}}$. Then we claim that $\rho_{\mathrm{reg}}(\alpha) \colon \mathbb{C}[G] \to \mathbb{C}[G]$ is left multiplication by α:

$$\rho_{\mathrm{reg}}(\alpha)(\beta) = \alpha\beta.$$

To see this, first suppose that $\alpha = g$ and that $\beta = \sum_{h \in G} s_h \cdot h$. Then

$$\rho_V(g)(\beta) = \rho_V(g)\left(\sum_{h \in G} s_h \cdot h\right) = \sum_{h \in G} s_h \cdot (gh),$$

by the definition of the regular representation. Thus $\rho_V(g)(\beta) = g\cdot\beta$ by the definition of multiplication in $\mathbb{C}[G]$. The case where $\alpha = \sum_{g\in G} t_g \cdot g$ then follows since

$$\begin{aligned}\rho_{\mathrm{reg}}(\alpha)(\beta) &= \sum_{g\in G} t_g \rho_V(g)(\beta) = \sum_{g\in G} t_g(g\cdot\beta)\\ &= \left(\sum_{g\in G} t_g\cdot g\right)\cdot\beta = \alpha\cdot\beta,\end{aligned}$$

since multiplication in $\mathbb{C}[G]$ distributes over addition.

For a finite group G, let $V_1,\dots,V_h$ denote the distinct irreducible representations of G up to isomorphism. If $d_i = \dim V_i$, then $\operatorname{End} V_i \cong \mathbb{M}_{d_i}(\mathbb{C})$ after we have chosen a basis. For each i, we have the $\mathbb{C}$-algebra homomorphism $\rho_{V_i}\colon \mathbb{C}[G] \to \operatorname{End} V_i$ and hence the $\mathbb{C}$-algebra homomorphism

$$\rho = (\rho_{V_1},\dots,\rho_{V_h})\colon \mathbb{C}[G] \to \operatorname{End} V_1 \times\cdots\times \operatorname{End} V_h \cong \mathbb{M}_{d_1}(\mathbb{C})\times\cdots\times\mathbb{M}_{d_h}(\mathbb{C}),$$

where the above isomorphism is of $\mathbb{C}$-algebras (and the products are given the product algebra structure as described in Example 1.2 (3). Viewing $\mathbb{C}[G]$ as $L^2(G)$, we denote the image $\rho(f)$ of f by $\hat{f}$ and call it the *Fourier transform* of f, for reasons which we will explain later. Note that $\widehat{f_1 * f_2} = \hat{f}_1\hat{f}_2$, which just says that ρ is an algebra homomorphism.

Theorem 1.7 (Wedderburn)**.** *The map ρ is an isomorphism. In particular, as $\mathbb{C}$-algebras,*

$$\mathbb{C}[G] \cong \mathbb{M}_{d_1}(\mathbb{C})\times\cdots\times\mathbb{M}_{d_h}(\mathbb{C}).$$

Proof. First, since ρ is a homomorphism, it suffices to show that it is a bijection. Next,

$$\dim\mathbb{C}[G] = \#(G) = \sum_{i=1}^{h} d_i^2 = \dim(\mathbb{M}_{d_1}(\mathbb{C})\times\cdots\times\mathbb{M}_{d_h}(\mathbb{C})).$$

As ρ is a linear map between two finite dimensional vector spaces of the same dimension, ρ is a bijection $\iff$ ρ is injective $\iff$ $\operatorname{Ker}\rho = 0$.

Thus assume that $\rho(\alpha) = 0$. We must show that $\alpha = 0$. By definition, $\rho_{V_i}(\alpha) = 0$ for every irreducible representation V_i. Using (3) of Remark 1.5, it then follows that $\rho_V(\alpha) = 0$ for every representation V. In particular, taking $V = \mathbb{C}[G]$, viewed as the regular representation, it follows that $\rho_{\mathrm{reg}}(\alpha) = 0$. By Example 1.6(2), this says that multiplication by α on $\mathbb{C}[G]$

is identically 0, i.e. $\alpha \cdot \beta = 0$ for all $\beta \in \mathbb{C}[G]$. Taking $\beta = 1$, we see that $0 = \alpha \cdot 1 = \alpha$. Hence $\alpha = 0$. It follows that ρ is injective and thus an isomorphism. $\square$

Remark 1.8. If k has characteristic zero but is not necessarily algebraically closed, then one can show that, as k-algebras,

$$k[G] \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_k}(D_k),$$

where the D_k are division algebras, possibly fields, containing k. For example,

$$\mathbb{Q}[\mathbb{Z}/n\mathbb{Z}] \cong \mathbb{Q} \times \mathbb{Q}(e^{2\pi i/n}).$$

It is also possible for non-commutative division algebras to appear. For example, if Q is the quaternion group, then

$$\mathbb{R}[Q] \cong \mathbb{R}^4 \times \mathbb{H}.$$

Next, we relate the isomorphism in Wedderburn's theorem to the center of $\mathbb{C}[G]$. We have stated (without proof) that the center of $\mathbb{M}_d(\mathbb{C})$ is $\mathbb{C} \cdot \mathrm{Id}$. Thus the center of $\mathbb{M}_{d_1}(\mathbb{C}) \times \cdots \times \mathbb{M}_{d_h}(\mathbb{C})$ is $\mathbb{C} \cdot \mathrm{Id} \times \cdots \times \mathbb{C} \cdot \mathrm{Id}$. As for $\mathbb{C}[G]$, it is a little easier to describe its center using the incarnation $\mathbb{C}[G] \cong L^2(G)$.

Proposition 1.9. *The center of $L^2(G)$ under the operation of convolution is the vector subspace Z of class functions.*

Proof. Since $\{\delta_x : x \in G\}$ is a basis for $L^2(G)$, a function $f \in L^2(G)$ is in the center of $L^2(G) \iff$ for all $x \in G$, $\delta_x * f = f * \delta_x \iff$ for all $x \in G$ and all $g \in G$, $\delta_x * f(g) = f * \delta_x(g)$. We have seen in the HW that $\delta_x * f(g) = f(x^{-1}g)$ and that $f * \delta_x(g) = f(gx^{-1})$. Thus f is in the center of $L^2(G) \iff$ for all $x, g \in G$, $f(x^{-1}g) = f(gx^{-1}) \iff$ for all $x, g \in G$, $f(xg) = f(gx)$ (replacing x^{-1} by x) $\iff f$ is a class function. $\square$

Via the isomorphism ρ, the center of $\mathbb{C}[G]$ has to correspond to the center of $\mathbb{M}_{d_1}(\mathbb{C}) \times \cdots \times \mathbb{M}_{d_h}(\mathbb{C})$. In fact, we have already computed the image $\rho(f)$ of a class function f, in Proposition 1.3 of the handout "Characters II:"

$$\rho(f) = (t_1 \,\mathrm{Id}, \ldots, t_h \,\mathrm{Id}),$$

where $t_i = \dfrac{\#(G)\langle f, \overline{\chi}_{V_i}\rangle}{d_i}$.

To conclude this section, we give a formula for ρ^{-1}:

Proposition 1.10 (Fourier inversion). *Given* $(A_1, \dots, A_h) \in \operatorname{End} V_1 \times \cdots \times \operatorname{End} V_h \cong \mathbb{M}_{d_1}(\mathbb{C}) \times \cdots \times \mathbb{M}_{d_h}(\mathbb{C})$, $\rho^{-1}(A_1, \dots, A_h) = \sum_g t_g \cdot g$, *where*

$$t_g = \frac{1}{\#(G)} \sum_{i=1}^{h} d_i \operatorname{Tr}(\rho_{V_i}(g^{-1})A_i).$$

Proof. By linearity, it is enough to check this formula for $(A_1, \dots, A_h) = \rho(x) = \rho(\delta_x)$, identifying the basis vector $x \in \mathbb{C}[G]$ with the basis element $\delta_x \in L^2(G)$. In other words, we can take $A_i = \rho_{V_i}(x)$. Then

$$\operatorname{Tr}(\rho_{V_i}(g^{-1})A_i) = \operatorname{Tr}(\rho_{V_i}(g^{-1})\rho_{V_i}(x)) = \operatorname{Tr}(\rho_{V_i}(g^{-1}x)) = \chi_{V_i}(g^{-1}x),$$

and so we want to show that $t_g = 1$ if $g = x$ and $t_g = 0$ otherwise, where

$$t_g = \frac{1}{\#(G)} \sum_{i=1}^{h} d_i \chi_{V_i}(g^{-1}x).$$

But as $\sum_{i=1}^{h} d_i \chi_{V_i} = \chi_{\mathrm{reg}}$ is the character of the regular representation,

$$\sum_{i=1}^{h} d_i \chi_{V_i}(g^{-1}x) = \begin{cases} \#(G), & \text{if } g^{-1}x = 1; \\ 0, & \text{otherwise.} \end{cases}$$

This implies that $t_g = 1$ if $g^{-1}x = 1$, i.e. $g = x$, and $t_g = 0$ otherwise, as claimed. □

2 A basis for $L^2(G)$

We have seen that the characters of the distinct irreducible representations are a unitary basis for the space of class functions. It is natural to ask if we can use representation theory to find a basis for all of $L^2(G)$. We shall outline how to do so.

Lemma 2.1. *Let V and W be two irreducible G-representations and let $F\colon V \to W$ be a linear map. Define*

$$p(F) = \frac{1}{\#(G)} \sum_{g \in G} \rho_W(g) \circ F \circ \rho_V(g)^{-1}.$$

Then:

(i) *If V and W are not isomorphic, then $p(F) = 0$.*

(ii) *If* $V = W$*, then*

$$p(F) = \frac{\operatorname{Tr} F}{\dim V} \operatorname{Id}.$$

Proof. (i) We have seen that p is a projection onto $\operatorname{Hom}^G(V, W)$. But if V and W are not isomorphic, then $\operatorname{Hom}^G(V, W) = 0$ by Schur's lemma. Thus $p(F) = 0$.

(ii) Again by Schur's lemma, if V is irreducible, then $\operatorname{Hom}^G(V, W) \cong \mathbb{C} \cdot \operatorname{Id}$. Thus $p(F) = t \operatorname{Id}$ for some $t \in \mathbb{C}$. Taking the trace, we see that

$$\operatorname{Tr}(p(F)) = \operatorname{Tr}(t \operatorname{Id}) = t \dim V.$$

On the other hand,

$$\operatorname{Tr}(p(F)) = \frac{1}{\#(G)} \sum_{g \in G} \operatorname{Tr}(\rho_V(g) \circ F \circ \rho_V(g)^{-1}) = \frac{1}{\#(G)} \sum_{g \in G} \operatorname{Tr} F,$$

using the identity that $\operatorname{Tr}(ABA^{-1}) = \operatorname{Tr} B$ for every invertible matrix A. Thus $\operatorname{Tr}(p(F)) = \operatorname{Tr} F$. Comparing this with $\operatorname{Tr}(p(F)) = t \dim V$ gives $t = \operatorname{Tr} F / \dim V$, which is the formula of (ii). □

We now interpret the lemma in terms of the matrix coefficients of $\rho_V(g)$ and $\rho_W(g)$:

Corollary 2.2. *Let* V *and* W *be two irreducible* G*-representations and suppose that* $v_1, \ldots, v_d$ *is a basis for* V *and* $w_1, \ldots, w_e$ *is a basis for* W*. For* $g \in G$*, let* $\rho_V(g)_{ij}$ *be the* $(i, j)^{th}$ *entry in the matrix for* $\rho_V(g)$ *corresponding to the basis* $v_1, \ldots, v_d$*, and similarly for* $\rho_W(g)_{ij}$*. Then*

(i) *If* V *and* W *are not isomorphic, then, for all* i, j*,* $1 \le i, j \le d$ *and all* k, ℓ*,* $1 \le k, \ell \le e$*,*

$$\frac{1}{\#(G)} \sum_{g \in G} \rho_V(g^{-1})_{ij} \rho_W(g)_{k\ell} = 0.$$

(ii) *If* $V = W$ *and* $v_i = w_i$ *for all* i*, then for all* i, j, k, ℓ*,* $1 \le i, j, k, \ell \le d$*,*

$$\frac{1}{\#(G)} \sum_{g \in G} \rho_V(g^{-1})_{ij} \rho_V(g)_{k\ell} = \begin{cases} \frac{1}{\dim V}, & \text{if } i = \ell \text{ and } j = k; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $F_{rs}\colon V \to W$ be the linear map defined by $F_{rs}(v_r) = w_s$ and $F_{rs}(v_i) = 0$, $i \neq r$. Then a computation shows that

$$\rho_W(g) \circ F_{rs} \circ \rho_V(g)^{-1}(v_i) = \sum_{\ell=1}^{e} \rho_V(g^{-1})_{ri}\rho_W(g)_{\ell s} w_\ell.$$

Hence, summing over all $g \in G$ and dividing by $\#(G)$, we see that

$$p(F_{rs})(v_i) = \sum_{\ell=1}^{e} \left(\frac{1}{\#(G)} \sum_{g\in G} \rho_V(g^{-1})_{ri}\rho_W(g)_{\ell s} \right) w_\ell.$$

If V and W are not isomorphic, then, for all r, s, $p(F_{rs}) = 0$, so $p(F_{rs})(v_i) = 0$ for all i. This says that, for all r, s, i, ℓ, the coefficient of w_ℓ in $p(F_{rs})(v_i)$ is 0, which is (i) (with a different labeling of the indices). As for (ii), we know that $p(F_{rs})$ is of the form $t \cdot \mathrm{Id}$, in particular it only has nonzero entries along the diagonal. Moreover, the diagonal entry for v_i in $\rho_V(g) \circ F_{rs} \circ \rho_V(g)^{-1}(v_i)$ is $\rho_V(g^{-1})_{ri}\rho_V(g)_{is}$. Again, summing over $g \in G$ and dividing by $\#(G)$, we see that

$$\frac{1}{\#(G)} \sum_{g\in G} \rho_V(g^{-1})_{ri}\rho_V(g)_{js} = \begin{cases} \frac{\operatorname{Tr} F_{rs}}{\dim V}, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\operatorname{Tr} F_{rs} = 0$ if $r \neq s$ and $\operatorname{Tr} F_{rr} = 1$, we get the formula in (ii). □

The appearance of the term $\rho_V(g^{-1})_{ij}$ is hard to exploit, since in general there is no good formula for $\rho_V(g^{-1})$ in terms of $\rho_V(g)$. However, if $\rho_V(g)$ is unitary with respect to the basis $v_1, \dots, v_d$, then things are much better:

$$\rho_V(g^{-1}) = \rho_V(g)^{-1} = {}^*\rho_V(g)$$

is the adjoint matrix, and hence

$$\rho_V(g^{-1})_{ij} = \overline{\rho_V(g)}_{ji}.$$

Thus

$$\frac{1}{\#(G)} \sum_{g\in G} \rho_V(g^{-1})_{ij}\rho_W(g)_{k\ell} = \frac{1}{\#(G)} \sum_{g\in G} \overline{\rho_V(g^{-1})}_{ji}\rho_W(g)_{k\ell} = \langle (\rho_W)_{k\ell}, (\rho_V)_{ji} \rangle.$$

In this case, the formulas of (i) and (ii) above read:

(i) If V and W are not isomorphic, then, for all i, j, $1 \leq i, j \leq d$ and all k, ℓ, $1 \leq k, \ell \leq e$,

$$\langle (\rho_W)_{k\ell}, (\rho_V)_{ji} \rangle = 0.$$

(ii) If $V = W$ and $v_i = w_i$ for all i, then for all i, j, k, ℓ, $1 \leq i, j, k, \ell \leq d$,

$$\langle (\rho_V)_{k\ell}, (\rho_V)_{ji} \rangle = \begin{cases} \frac{1}{\dim V}, & \text{if } i = \ell \text{ and } j = k; \\ 0, & \text{otherwise.} \end{cases}$$

Summarizing, we obtain:

Theorem 2.3. *Let $V_1, \ldots, V_h$ be the distinct irreducible representations of G up to isomorphism and let $d_i = \dim V_i$. We suppose that, for each i, we have chosen a G-invariant Hermitian inner product on V_i and a unitary basis $v_1, \ldots, v_{d_i}$ for this inner product and let $(\rho_{V_i}(g)_{rs})$ be the (unitary) matrix for $\rho_{V_i}(g)$ with respect to this basis. For each i, $1 \leq i \leq h$ and for r, s with $1 \leq r, s \leq d_i$, set*

$$f_{i,r,s}(g) = \sqrt{d_i} \rho_{V_i}(g)_{rs}.$$

Then the normalized matrix coefficients $f_{i,r,s}(g)$ are a basis for $L^2(G)$.

Proof. The calculations above show that the functions $f_{i,r,s}$ are orthonormal, in the sense that $\langle f_{i,r,s}, f_{j,t,u} \rangle = 0$ unless $i = j$, $r = t$, $s = u$, and $\langle f_{i,r,s}, f_{i,r,s} \rangle = 1$. In particular they are linearly independent. But the number of such functions is $\sum_{i=1}^{h} d_i^2 = \#(G)$, and so they must be a basis for $L^2(G)$. ☐

Remark 2.4. For every representation V of G, a G-invariant positive definite Hermitian inner product always exists on V: choose an arbitrary positive definite Hermitian inner product H_0 on V and average over G, i.e. set

$$H(v, w) = \frac{1}{\#(G)} \sum_{g \in G} H_0(\rho_V(g)(v), \rho_V(g)(w)).$$

It is clear that H is a positive definite Hermitian inner product, and the usual arguments show that H is G-invariant, i.e. that

$$H(\rho_V(g)(v), \rho_V(g)(w)) = H(v, w)$$

for all $v, w \in V$ and $g \in G$. Thus the matrices for $\rho_V(g)$ with respect to a unitary basis are unitary.

The G-invariant positive definite Hermitian inner product H is not necessarily unique. However, if V is irreducible, then an argument with Schur's lemma shows that every other G-invariant positive definite Hermitian inner product H' is of the form tH for some positive real number t. However, we omit the details.

3 The Fourier transform for finite abelian groups

In this section, we assume that G is a finite **abelian** group.

Definition 3.1. The *dual group* $\widehat{G}$ is the set of homomorphisms $\lambda\colon G \to \mathbb{C}^*$. Thus in particular $\widehat{G} \subseteq L^2(G)$. It is easy to check that (for an arbitrary, not necessarily abelian group G) that $\widehat{G}$ is a group under pointwise multiplication of homomorphisms, i.e. if we define the product $\lambda_1\lambda_2$ by

$$(\lambda_1\lambda_2)(g) = \lambda_1(g)\lambda_2(g).$$

The multiplicative inverse of λ is $\lambda^{-1} = 1/\lambda$ (**not** the inverse function!), which is again a homomorphism from G to $\mathbb{C}^*$. Note that, as $\lambda(g)$ has finite order, $\lambda(g)$ has absolute value one, and hence $\lambda^{-1} = \overline{\lambda}$.

Beginning with the next lemma, however, we strongly use the fact that G is abelian.

Lemma 3.2. $\#(\widehat{G}) = \#(G)$. *Moreover, the* $\lambda \in \widehat{G}$ *are a unitary basis of* $L^2(G)$ *with respect to the Hermitian inner product.*

Proof. For a finite abelian group G, if $V_1, \dots, V_h$ are the irreducible representations, with $d_i = \dim V_i$, then we have seen that $d_i = 1$ for all i and that $h = \#(G)$. Then the V_i are necessarily of the form $\mathbb{C}(\lambda_i)$, $\lambda_i \in \widehat{G}$ and each element of $\widehat{G}$ appears exactly once as a λ_i. Thus $h = \#(\widehat{G}) = \#(G)$.

To see the final statement, we know that, for a general finite group G, the characters are a basis for the space of class functions. For an abelian group, a character is just an element of $\widehat{G}$ and a class function is just a function, so that $\widehat{G}$ is a basis of $L^2(G)$. It is a unitary basis by the orthogonality relations for characters (or by an easy direct argument in this case): $\langle \mu, \lambda \rangle = 1$ if $\lambda = \mu$ and $\langle \mu, \lambda \rangle = 0$ otherwise. □

Example 3.3. For $G = \mathbb{Z}/n\mathbb{Z}$, every homomorphism $\lambda\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$ is of the form λ_a (here a is an integer mod n), where

$$\lambda_a(k) = e^{2\pi i a k/n}.$$

In particular, $\lambda_a(1) = e^{2\pi i a/n}$ is an element of μ_n, ie. an element of $\mathbb{C}^*$ of order n, which determines and is determined by the homomorphism. Also, by the rules of exponents $\lambda_a \cdot \lambda_b = \lambda_{a+b}$, and $\lambda_a = 1 \iff a = 0$ as an element of $\mathbb{Z}/n\mathbb{Z}$. Thus $a \mapsto \lambda_a$ is an isomorphism from $\mathbb{Z}/n\mathbb{Z}$ to $\widehat{\mathbb{Z}/n\mathbb{Z}}$.

More generally, every finite abelian group G is isomorphic to $\widehat{G}$, but there is no "natural" choice of isomorphism.

Definition 3.4. For a finite abelian group G, and a function $f \in L^2(G)$, we define the *Fourier transform* $\hat{f} \in L^2(\widehat{G})$ by:

$$\hat{f}(\lambda) = \sum_{g \in G} f(g)\overline{\lambda(g)} = \#(G)\langle f, \lambda\rangle.$$

Thus the Fourier transform is a linear map $FT\colon L^2(G) \to L^2(\widehat{G})$.

Remark 3.5. Other normalizations are also possible. For example, one could define $\hat{f}(\lambda)$ to be $\langle f, \lambda\rangle$ or $\langle f, \bar{\lambda}\rangle$, with minor changes in the formulas below. In fact, we will use a different normalization in the nonabelian case.

The main point in what follows is that there are **two** different and interesting bases for $L^2(G)$. The first is $\{\delta_x : x \in X\}$. This is almost but not quite unitary with respect to the Hermitian inner product on $L^2(G)$: in fact,

$$\langle \delta_x, \delta_y\rangle = \begin{cases} 0, & \text{if } x \neq y; \\ \frac{1}{\#(G)}, & \text{if } x = y. \end{cases}$$

The second basis is the unitary basis $\widehat{G}$. Given $f \in L^2(G)$, the coefficient of f with respect to the basis element δ_x for the basis $\{\delta_x : x \in X\}$ is by definition $f(x)$. The coefficient of f with respect to the basis element λ for the unitary basis $\widehat{G}$ is

$$\langle f, \lambda\rangle = \frac{1}{\#(G)}\hat{f}(\lambda).$$

In much of what follows, the arguments will boil down to comparing these two different descriptions of a function f.

Example 3.6. (1) For $G = \mathbb{Z}/n\mathbb{Z}$, and using the remarks above to identify $\lambda_a \in \widehat{\mathbb{Z}/n\mathbb{Z}}$ with $a \in \mathbb{Z}/n\mathbb{Z}$, we have

$$\hat{f}(a) = \sum_{k=0}^{n-1} f(k)e^{-2\pi i ak/n}.$$

(2) For a general abelian group G and $x \in G$, we have

$$\hat{\delta}_x(\lambda) = \overline{\lambda(x)} = \lambda^{-1}(x).$$

Thus $\hat{\delta}_x = \mathrm{ev}_x \circ \sigma = \overline{\mathrm{ev}}_x$, where $\mathrm{ev}_x \in L^2(\widehat{G})$ is evaluation at x and $\sigma\colon \widehat{G} \to \widehat{G}$ is complex conjugation of homomorphisms.

(3) Since $\widehat{G} \subseteq L^2(G)$, we can also form the Fourier transform $\hat{\mu}$ of a $\mu \in \widehat{G}$.

Claim 3.7. $\hat{\mu} = \#(G)\delta_\mu$.

Proof. By definition, $\hat{\mu}(\lambda) = \#(G)\langle \mu, \lambda\rangle$. But $\langle \mu, \lambda\rangle = \delta_\mu(\lambda)$, so that $\hat{\mu} = \#(G)\delta_\mu$. □

Theorem 3.8. *For all $f, f_1, f_2 \in L^2(G)$,*

(i) $\boxed{f = \dfrac{1}{\#(G)} \sum_{\lambda \in \widehat{G}} \hat{f}(\lambda)\lambda}$ (Fourier inversion)

(ii) $\boxed{\langle f_1, f_2\rangle = \dfrac{1}{\#(G)} \langle \hat{f}_1, \hat{f}_2\rangle}$ (Plancherel formula)

(iii) $\boxed{\widehat{f_1 * f_2} = \hat{f}_1 \hat{f}_2}$ *and* $\boxed{\widehat{f_1 f_2} = \dfrac{1}{\#(G)}(\hat{f}_1 * \hat{f}_2)}$

Proof. (i) Since $\widehat{G}$ is a unitary basis for $L^2(G)$,

$$f = \sum_{\lambda \in \widehat{G}} \langle f, \lambda\rangle\lambda = \frac{1}{\#(G)} \sum_{\lambda \in \widehat{G}} \hat{f}(\lambda)\lambda.$$

(ii) Again using the fact that $\widehat{G}$ is a unitary basis for $L^2(G)$,

$$\begin{aligned}\langle f_1, f_2\rangle &= \sum_{\lambda \in \widehat{G}} \langle f_1, \lambda\rangle \overline{\langle f_1, \lambda\rangle} = \frac{1}{\#(G)^2} \sum_{\lambda \in \widehat{G}} \hat{f}_1(\lambda)\overline{\hat{f}_2(\lambda)} \\ &= \frac{1}{\#(G)} \langle \hat{f}_1, \hat{f}_2\rangle.\end{aligned}$$

(iii) In fact, we have essentially proved the first formula, see (2) of Remark 1.5. The point is that, in the abelian case, we have defined $\rho\colon L^2(G) \to \mathbb{C}^h$, where $h = \#(G)$, by setting $\rho_{\mathbb{C}(\lambda)}(f) = \sum_{g \in G} f(g)\lambda(g) = \#(G)\langle f, \overline{\lambda}\rangle$. Thus $\rho_{\mathbb{C}(\lambda)}(f) = \hat{f}(\overline{\lambda})$. By Remark 1.5, $\rho_{\mathbb{C}(\lambda)}(f_1 * f_2) = \rho_{\mathbb{C}(\lambda)}(f_1)\rho_{\mathbb{C}(\lambda)}(f_2)$, and this proves the formula up to conjugating λ.

It is however easy to give a direct proof. It suffices by linearity to check the formula for $f_1 = \delta_x$ and $f_2 = f$ an arbitrary element of $L^2(G)$, since the δ_x are a basis for $L^2(G)$. Recall that $(\delta_x * f)(g) = f(x^{-1}g)$. Then

$$\begin{aligned}\widehat{\delta_x * f}(\lambda) &= \sum_{g\in G} f(x^{-1}g)\overline{\lambda(g)} = \sum_{g\in G} f(g)\overline{\lambda(xg)} \\ &= \overline{\lambda(x)}\sum_{g\in G} f(g)\overline{\lambda(g)} = \overline{\lambda(x)}\hat{f}(\lambda).\end{aligned}$$

But $\overline{\lambda(x)} = \hat{\delta}_x(\lambda)$, by (2) of Example 3.6, and so

$$\widehat{\delta_x * f} = \hat{\delta}_x \hat{f}$$

as claimed.

To prove the second formula in (iii), it is enough to check it for $f_1 = \mu \in \widehat{G} \subseteq L^2(G)$ and $f_2 = f$ arbitrary, using the fact that $\widehat{G}$ is a basis for $L^2(G)$. Here $\mu f(g) = \mu(g)f(g)$, so that

$$\begin{aligned}\widehat{\mu f}(\lambda) &= \sum_{g\in G} \mu(g)f(g)\overline{\lambda(g)} = \sum_{g\in G} f(g)\overline{(\mu^{-1}\lambda)(g)} \\ &= \hat{f}(\mu^{-1}\lambda) = \delta_\mu * \hat{f}.\end{aligned}$$

By (3) of Example 3.6, $\hat{\mu} = \#(G)\delta_\mu$. Thus

$$\widehat{\mu f} = \frac{1}{\#(G)}\hat{\mu} * \hat{f}$$

as claimed. □

We give another interpretation of Fourier inversion as follows. Since $\widehat{G}$ is a finite abelian group, we can consider its dual group $\widehat{\widehat{G}}$. By a homework problem, we have the homomorphism $\mathrm{ev}\colon G \to \widehat{\widehat{G}}$ defined by $\mathrm{ev}(g)(\lambda) = \lambda(g)$, and it is an isomorphism. Thus, we can view $L^2(\widehat{\widehat{G}})$ as $L^2(G)$ and must compute the value $\hat{\hat{f}}$ on $g \in G$.

Proposition 3.9. $\hat{\hat{f}}(g) = \#(G)f(g^{-1})$.

Proof. By definition of the Fourier transform,

$$\begin{aligned}\hat{\hat{f}}(g) &= \sum_{\lambda\in\widehat{G}} \hat{f}(\lambda)\overline{\mathrm{ev}(g)(\lambda)} = \sum_{\lambda\in\widehat{G}} \hat{f}(\lambda)\overline{\lambda(g)} \\ &= \sum_{\lambda\in\widehat{G}}\sum_{h\in G} f(h)\overline{\lambda(h)\lambda(g)} = \sum_{h\in G} f(h)\left(\sum_{\lambda\in\widehat{G}}\overline{\lambda(hg)}\right).\end{aligned}$$

But the sum over all $\lambda \in \widehat{G}$ of $\overline{\lambda(gh)} = \lambda^{-1}(gh)$ is the same as the sum over all λ of $\lambda(gh)$, so that

$$\sum_{\lambda\in\widehat{G}} \overline{\lambda(hg)} = \sum_{\lambda\in\widehat{G}} \lambda(hg) = \begin{cases} \#(G), & \text{if } gh = 1, \text{ i.e. } h = g^{-1}; \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\hat{\hat{f}}(g) = f(g^{-1})\#(G)$. □

4 The non-abelian case

We will now reinterpret the results of Section 1 in the language of the previous section. For a finite group G, choose a set of irreducible representations $V_1, \dots, V_h$ of V in the usual way and set $\dim V_i = d_i$. We will think of the set $\{V_1, \dots, V_h\}$ as the set of irreducible representations of G up to isomorphism, and will denote this set by $\widehat{G}$. Note that, for a nonabelian G, $\widehat{G}$ is just a **set**, not a group, and there is no set $\widehat{\widehat{G}}$. We have defined an isomorphism

$$\rho = (\rho_{V_1}, \dots, \rho_{V_h})\colon \mathbb{C}[G] \to \operatorname{End} V_1 \times \cdots \times \operatorname{End} V_h \cong \mathbb{M}_{d_1}(\mathbb{C}) \times \cdots \times \mathbb{M}_{d_h}(\mathbb{C}),$$

and will view this rather as an isomorphism from $L^2(G)$ to $\operatorname{End} V_1 \times \cdots \times \operatorname{End} V_h$. The i^{th} component of $\rho(f)$ is then

$$\rho_{V_i}(f) = F_{V_i,f} = \sum_{g\in G} f(g)\rho_{V_i}(g).$$

We think of this as defining a "function" $\hat{f}$ whose value at V_i is the linear map $\rho_{V_i}(f) = F_{V_i,f}\colon V_i \to V_i$.

This construction differs from the Fourier transform of an abelian group in two ways: First, in the abelian case, V_i is one-dimensional and thus $\operatorname{End} V_i$ can be identified with $\mathbb{C}$, and we can identify the set $\{V_1, \dots, V_h\}$ with $\widehat{G}$. Thus $\hat{f}$ is defined on $\widehat{G}$ and it has a well-defined value in $\mathbb{C}$, so it is just a function, i.e. an element of $L^2(\widehat{G})$. Second, we used the normalization $\hat{f}(\lambda) = \sum_{g\in G} f(g)\overline{\lambda(g)}$, so the above definition defines what we had previously defined to be $\hat{f}(\lambda^{-1})$, not $\hat{f}(\lambda)$. This is one of many annoying normalization issues, but we will not try to be consistent here.

Finally, we will define the adjoint ${}^*A$ of an $A \in \operatorname{End} V_i \cong \mathbb{M}_{d_i}(\mathbb{C})$ by taking the adjoint with respect to some G-invariant positive definite Hermitian inner product H on V_i, i.e. ${}^*A$ is defined by the property that

$$H(Av, w) = H(v, {}^*Aw)$$

for all $v, w \in V_i$. As in the discussion in Remark 2.4, such an H exists and is unique up to multiplication by a positive real number, and the adjoint is the same for all possible choices of H. In particular, since $\rho_{V_i}(g)$ is unitary with respect to H, we have ${}^*\rho_{V_i}(g) = \rho_{V_i}(g)^{-1}$.

With this said, we have the non-abelian analogue of Theorem 3.8:

Theorem 4.1. *For all $f, f_1, f_2 \in L^2(G)$,*

(i) $$\boxed{f = \frac{1}{\#(G)} \sum_{i=1}^{h} d_i \operatorname{Tr}(\rho_{V_i}(g)^{-1} \hat{f}(V_i))} \qquad \text{(Fourier inversion)}$$

(ii) $$\boxed{\langle f_1, f_2 \rangle = \frac{1}{\#(G)^2} \sum_{i=1}^{h} d_i \operatorname{Tr}(\hat{f}_1(V_i) \cdot {}^*(\hat{f}_1(V_i))} \qquad \text{(Plancherel formula)}$$

(iii) $$\boxed{\widehat{f_1 * f_2} = \hat{f}_1 \hat{f}_2}$$

Proof. We have proved (i) (in a slightly different notation) in Proposition 1.10. And (iii) follows from Lemma 1.4 (see also (2) of Remark 1.5). So we must show (ii). Since $\{\delta_x : x \in G\}$ is a basis for $L^2(G)$, it is enough to check (ii), using the bilinearity of both sides, for $f_1 = \delta_x$ and $f_2 = \delta_y$, for all $x, y \in G$. In this case,

$$\langle \delta_x, \delta_y \rangle = \begin{cases} \frac{1}{\#(G)}, & \text{if } x = y; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, by definition, $\hat{\delta}_x(V_i) = \rho_{V_i}(x)$. Thus, the right hand side of (ii) is equal to

$$\begin{aligned} \frac{1}{\#(G)^2} \sum_{i=1}^{h} d_i \operatorname{Tr}(\rho_{V_i}(x) \cdot {}^*\rho_{V_i}(y)) &= \frac{1}{\#(G)^2} \sum_{i=1}^{h} d_i \operatorname{Tr}(\rho_{V_i}(x)\rho_{V_i}(y)^{-1}) \\ &= \frac{1}{\#(G)^2} \sum_{i=1}^{h} d_i \operatorname{Tr}(\rho_{V_i}(xy^{-1})) \\ &= \frac{1}{\#(G)^2} \sum_{i=1}^{h} d_i \chi_{V_i}(xy^{-1}) = \frac{1}{\#(G)^2} \chi_{\mathrm{reg}}(xy^{-1}). \end{aligned}$$

But $\chi_{\mathrm{reg}}(xy^{-1}) = \#(G)$ if $x = y$ and $\chi_{\mathrm{reg}}(xy^{-1}) = 0$ otherwise. Thus we see that the right hand side of (ii) is equal to $\langle \delta_x, \delta_y \rangle$ as claimed. □