Characters I

Throughout, G denotes a **finite** group.

1 The character of a representation

Definition 1.1. Let V (or ρ_V) be a G-representation. Then the *character* χ_V (or χ_{ρ_V}) of V is the function $\chi_V : G \to \mathbb{C}$ defined by:

$$\chi_V(g) = \operatorname{Tr} \rho_V(g).$$

Note that, for all $g \in G$, $\chi_V(g)$ is a sum of roots of unity.

Example 1.2.

1. If V is the trivial representation (i.e. $\dim V = 1$ and $\rho_V(g) = \mathrm{Id}$ for all $g \in G$), then $\chi_V(g) = 1$ for all $g \in G$. We sometimes write χ_1 or just 1 for this character.

2. More generally, if $\dim V = 1$ and $\rho_V(g)(v) = \lambda(g)$, where $\lambda\colon G \to \mathbb{C}^*$ is a homomorphism, then $\chi_V = \lambda$. For example, for the one-dimensional representation V of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{C}$ for which $\lambda(k) = e^{2\pi ik/n}$, we have $\chi_V(k) = \lambda(k) = e^{2\pi ik/n}$.

3. The group D_n is generated by elements σ and τ, where σ is a counterclockwise rotation by the angle $2\pi k/n$ and τ is reflection in the x-axis. For the representation of D_n on $V = \mathbb{C}^2$ for which

$$\rho_V(\sigma^k) = A_{2\pi k/n} = \begin{pmatrix} \cos 2\pi k/n & -\sin 2\pi k/n \\ \sin 2\pi k/n & \cos 2\pi k/n \end{pmatrix};$$
$$\rho_V(\sigma^k\tau) = B_{2\pi k/n} = \begin{pmatrix} \cos 2\pi k/n & \sin 2\pi k/n \\ \sin 2\pi k/n & -\cos 2\pi k/n \end{pmatrix},$$

we clearly have:

$$\chi_V(\sigma^k) = 2\cos 2\pi k/n; \qquad \chi_V(\sigma^k\tau) = 0.$$

4. For the 2-dimensional representation V of the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ described previously, we have

$$\chi_V(1) = 2; \qquad \chi_V(-1) = -2; \qquad \chi_V(\pm i) = \chi_V(\pm j) = \chi_V(\pm k) = 0.$$

5. For the standard representation of S^n on $\mathbb{C}^n$, the corresponding character χ satisfies: $\chi(\sigma)$ is the number of i such that $\sigma(i) = i$. Hence, if $\sigma = \gamma_1 \cdots \gamma_k$ is a product of disjoint cycles γ_i of lengths $\ell_i > 1$, then $\chi(\sigma) = n - \sum_{i=1}^k \ell_i$.

6. More generally, if X is a G-set and $\rho_{\mathbb{C}[X]}$ is the corresponding permutation representation on $\mathbb{C}[X]$, with character $\chi_{\mathbb{C}[X]}$, then

$$\chi_{\mathbb{C}[X]}(g) = \#(X^g),$$

where X^g is the fixed set of g: $X^g = \{x \in X : g \cdot x = x\}$. In particular, if $X = G$, where G acts on itself by left multiplication, then $\mathbb{C}[G]$ is the regular representation. We write χ_{reg} for the character $\chi_{\mathbb{C}[G]}$. For the left multiplication action, given $g \in G$ and $x \in G$, g fixes x, i.e. $gx = x \iff g = 1$, and the element 1 fixes every $x \in G$. In other words, $G^g = \emptyset$ if $g \neq 1$ and $G^1 = G$. Thus:

$$\chi_{\mathrm{reg}}(g) = \begin{cases} \#(G), & \text{if } g = 1; \\ 0, & \text{if } g \neq 1. \end{cases}$$

We list some basic properties of characters.

1. If V is the trivial representation, then $\chi_V(g) = 1$ for all $g \in G$, i.e. χ_V is the constant function 1.

2. For every representation V,

$$\boxed{\chi_V(1) = \dim V = \deg \rho_V}.$$

This follows since $\rho_V(1) = \mathrm{Id}$ corresponds to the $d \times d$ identity matrix I, where $d = \dim V = \deg \rho_V$, and $\operatorname{Tr} I = d$.

3. For all $g, h \in G$,

$$\boxed{\chi_V(hgh^{-1}) = \chi_V(g)}.$$

This follows since by definition

$$\chi_V(hgh^{-1}) = \operatorname{Tr}(\rho_V(h) \circ \rho_V(g) \circ \rho_V(h)^{-1}) = \operatorname{Tr} \rho_V(g) = \chi_V(g).$$

4. For every $g \in G$,

$$\boxed{\chi_V(g^{-1}) = \overline{\chi_V(g)}}.$$

To see this, note that, for every $g \in G$, $\rho_V(g) \in \operatorname{Aut} V$ has finite order. Hence $\rho_V(g)$ is diagonalizable and its eigenvalues are roots of unity, in particular complex numbers of absolute value 1. By a homework problem,

$$\chi_V(g^{-1}) = \operatorname{Tr} \rho_V(g^{-1}) = \operatorname{Tr} \rho_V(g)^{-1} = \overline{\operatorname{Tr} \rho_V(g)} = \overline{\chi_V(g)}.$$

Next, we see how the character behaves with respect to the standard constructions of linear algebra: Suppose that V_1, V_2, and V are G-representations. Then:

1. $\boxed{\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}}$

   This is an immediate consequence of the formula $\operatorname{Tr}(F_1 \oplus F_2) = \operatorname{Tr} F_1 + \operatorname{Tr} F_2$. Aplying this inductively, we see that $\chi_{V_1 \oplus \cdots \oplus V_k} = \chi_{V_1} + \cdots + \chi_{V_k}$. Also, if we let $V^n = \underbrace{V \oplus \cdots \oplus V}_{n \text{ times}}$, then $\chi_{V^n} = n\chi_V$.

2. $\boxed{\chi_{V^*} = \overline{\chi_V}}$

   To see this, first recall that $\operatorname{Tr} F = \operatorname{Tr} F^*$. Now $\rho_{V^*}(g) = (\rho_V(g)^{-1})^*$, and hence

   $$\chi_{V^*}(g) = \operatorname{Tr}(\rho_{V^*}(g)) = \operatorname{Tr}((\rho_V(g)^{-1})^*) = \operatorname{Tr}(\rho_V(g)^{-1}) = \overline{\operatorname{Tr}(\rho_V(g))} = \overline{\chi_V(g)}.$$

3. $\boxed{\chi_{\operatorname{Hom}(V_1,V_2)} = \overline{\chi_{V_1}}\chi_{V_2}}$

   The argument for this is similar to the argument for (2): Suppose that $F_1 \in \operatorname{Hom}(V_1, V_1)$ and that $F_2 \in \operatorname{Hom}(V_2, V_2)$. Then $(F_2)_* \circ (F_1)^*$ is a linear map from $\operatorname{Hom}(V_1, V_2)$ to $\operatorname{Hom}(V_1, V_2)$. We have, by a homework problem,

   $$\operatorname{Tr}((F_2)_* \circ (F_1)^*) = (\operatorname{Tr} F_1)(\operatorname{Tr} F_2).$$

   By definition, $\rho_{\operatorname{Hom}(V_1,V_2)} = (\rho_{V_2})_* \circ (\rho_{V_2}^{-1})^*$. Thus,

   $$\begin{aligned}\chi_{\operatorname{Hom}(V_1,V_2)}(g) &= \operatorname{Tr} \rho_{\operatorname{Hom}(V_1,V_2)}(g)) = \operatorname{Tr}((\rho_{V_2}(g))_* \circ (\rho_{V_1}(g)^{-1})^*) \\ &= \operatorname{Tr}(\rho_{V_2}(g)) \operatorname{Tr}(\rho_{V_1}(g)^{-1})) = \overline{\chi_{V_1}(g)}\chi_{V_2}(g).\end{aligned}$$

4. $\boxed{\chi_{V_1 \otimes V_2} = \chi_{V_1}\chi_{V_2}}$

   This follows from the fact that $\operatorname{Tr}(F_1 \otimes F_2) = (\operatorname{Tr} F_1)(\operatorname{Tr} F_2)$.

In particular, we see that the sum, product, and complex conjugates of characters are characters.

2 Orthogonality relations

There are many identities involving characters which are called orthogonality relations. To begin, recall that, given a G-representation V, we have defined a projection map $p\colon V \to V^G$ by

$$p(v) = \frac{1}{\#(G)} \sum_{g \in G} \rho_V(g)(v).$$

We also know that $\operatorname{Tr} p = \dim V^G$ by general linear algebra results about traces. Ccomputing the trace of p in two different ways then gives

$$\boxed{\dim V^G = \frac{1}{\#(G)} \sum_{g \in G} \chi_V(g)}$$

Applying this formula to $\operatorname{Hom}(V_1, V_2)$ gives:

$$\boxed{\dim \operatorname{Hom}^G(V_1, V_2) = \frac{1}{\#(G)} \sum_{g \in G} \overline{\chi_{V_1}(g)} \chi_{V_2}(g) = \frac{1}{\#(G)} \sum_{g \in G} \chi_{V_1}(g) \overline{\chi_{V_2}(g)}}$$

Finally, if V_1 and V_2 are irreducible, and using Schur's lemma, this becomes:

Proposition 2.1. *If V_1 and V_2 are irreducible, then*

$$\boxed{\frac{1}{\#(G)} \sum_{g \in G} \chi_{V_1}(g) \overline{\chi_{V_2}(g)} = \dim \operatorname{Hom}^G(V_1, V_2) = \begin{cases} 1, & \text{if } V_1 \cong V_2; \\ 0, & \text{if } V_1 \text{ is not isomorphic to } V_2. \end{cases}}$$

It's convenient to introduce the G-invariant positive definite Hermitian inner product on the vector space $\mathbb{C}(G)$, viewed as the space $L^2(G)$ of functions $f\colon G \to \mathbb{C}$:

$$\langle f_1, f_2 \rangle = \frac{1}{\#(G)} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

Thus we can restate the above proposition as: If V_1 and V_2 are irreducible, then

$$\boxed{\langle \chi_{V_1}, \chi_{V_2} \rangle = \dim \operatorname{Hom}^G(V_1, V_2) = \begin{cases} 1, & \text{if } V_1 \cong V_2; \\ 0, & \text{if } V_1 \text{ is not isomorphic to } V_2 \end{cases}}$$

Corollary 2.2. *Write* $V \cong V_1^{m_1} \oplus \cdots \oplus V_k^{m_k}$*, where* V_i *is irreducible,* V_i *is not isomorphic to* V_j *if* $i \neq j$*, and* $V_i^{m_i}$ *is shorthand for the direct sum*

$$\underbrace{V_i \oplus \cdots \oplus V_i}_{m_i \text{ times}}.$$

Then

$$\boxed{\langle \chi_V, \chi_V \rangle = \sum_{i=1}^{k} m_i^2}$$

In particular, V *is irreducible* $\iff \langle \chi_V, \chi_V \rangle = 1$.

Proof. By our formulas, $\chi_V = \sum_{i=1}^{k} m_i \chi_{V_i}$. Then, expanding out the inner product gives

$$\langle \chi_V, \chi_V \rangle = \sum_{i,j} m_i m_j \langle \chi_{V_i}, \chi_{V_j} \rangle.$$

As $\langle \chi_{V_i}, \chi_{V_j} \rangle$ is 1 if $i = j$ and 0 otherwise, the sum becomes $\sum_i m_i^2$ as claimed. The final statement follows since, if the m_i are positive integers, then $\sum_{i=1}^{k} m_i^2 = 1 \iff k = 1$ and $m_1 = 1$, which clearly happens $\iff V$ is irreducible. $\square$

Corollary 2.3. *Write* $V \cong V_1^{m_1} \oplus \cdots \oplus V_k^{m_k}$ *as in the previous corollary. Let* W *be an irreducible representation. Then*

$$\boxed{\langle \chi_W, \chi_V \rangle = \begin{cases} m_i, & \text{if } W \cong V_i; \\ 0, & \text{if } W \text{ is not isomorphic to } V_i \text{ for any } i. \end{cases}}$$

Hence two representations V *and* V' *are isomorphic* $\iff \chi_V = \chi_{V'}$*. In other words:*

The character χ_V ***determines the representation*** V ***up to isomorphism.***

Proof. We have seen that $\chi_V = \sum_{i=1}^{k} m_i \chi_{V_i}$, and hence

$$\langle \chi_W, \chi_V \rangle = \sum_{i=1}^{k} m_i \langle \chi_W, \chi_{V_i} \rangle.$$

But $\langle \chi_W, \chi_{V_i} \rangle = 1 \iff W \cong V_i$, which can happen for at most one i by the assumption that V_i is not isomorphic to V_j if $i \neq j$. Hence $\langle \chi_W, \chi_V \rangle = m_i$ if $W \cong V_i$ and $\langle \chi_W, \chi_V \rangle = 0$ if W is not isomorphic to any V_i.

To see the final statement, clearly, if $V \cong V'$, then $\chi_V = \chi_{V'}$. Conversely, suppose that $\chi_V = \chi_{V'}$. Write $V \cong V_1^{m_1} \oplus \cdots \oplus V_k^{m_k}$ as above. Then

$$\langle \chi_{V_i}, \chi_{V'} \rangle = \langle \chi_{V_i}, \chi_V \rangle = m_i,$$

and $\langle \chi_W, \chi_{V'} \rangle = \langle \chi_W, \chi_V \rangle = 0$ if W is an irreducible representation not isomorphic to V_i for some i. Hence $V' \cong V_1^{m_1} \oplus \cdots \oplus V_k^{m_k}$, and thus $V' \cong V$. □

Definition 2.4. If V is a representation and W is an irreducible representation, we define the *multiplicity of W in V* to be the nonnegative integer $\langle \chi_W, \chi_V \rangle$.

3 The regular representation

Our goal now will be to apply the results of the previous section to the regular representation $\mathbb{C}[G]$, whose character $\chi_{\mathbb{C}[G]} = \chi_{\text{reg}}$ we have computed. In fact, $\chi_{\text{reg}}(1) = \#(G)$ and $\chi_{\text{reg}}(g) = 0$ if $g \neq 1$.

Proposition 3.1. *Let W be an irreducible representation. Then*

$$\boxed{\langle \chi_W, \chi_{\text{reg}} \rangle = \dim W}$$

Proof. By definition and the above remarks,

$$\langle \chi_W, \chi_{\text{reg}} \rangle = \frac{1}{\#(G)} \sum_{g \in G} \chi_W(g)\overline{\chi_{\text{reg}}(g)} = \frac{\chi_W(1) \cdot \#(G)}{\#(G)} = \dim W.$$

□

Corollary 3.2. *Write*

$$\mathbb{C}[G] \cong W_1^{d_1} \oplus \cdots \oplus W_h^{d_h},$$

where the W_i are irreducible and, for $i \neq j$, W_i is not isomorphic to W_j. Then:

(i) $d_i = \dim W_i$.

(ii) *Every irreducible representation of V is isomorphic to W_i for a unique i. In particular, there are only finitely many irreducible G-representations up to isomorphism.*

Proof. The first statement follows from the previous proposition and Corollary 2.3. The second follows similarly, since if W is an irreducible representation, then $\langle \chi_W, \chi_{\text{reg}} \rangle = \dim W > 0$ and hence $W \cong W_i$ for some i. $\square$

Corollary 3.3. *If $W_1, \dots, W_h$ are the finitely many distinct irreducible G-representations up to isomorphism and $d_i = \dim W_i$, then*

$$\boxed{\sum_{i=1}^{h} d_i^2 = \#(G)}$$

$$\boxed{\sum_{i=1}^{h} d_i \chi_{W_i}(g) = \begin{cases} \#(G), & \text{if } g = 1; \\ 0, & \text{if } g \neq 1. \end{cases}}$$

Proof. We prove the second identity first. Since $\mathbb{C}[G] \cong W_1^{d_1} \oplus \cdots \oplus W_h^{d_h}$,

$$\chi_{\text{reg}} = \sum_{i=1}^{h} d_i \chi_{W_i}.$$

The result then follows from our calculation of χ_{reg}. The first identity is then a consequence, since, for every i, $\chi_{W_i}(1) = \dim W_i = d_i$. $\square$