Characters II: Class functions

1 Class functions

Definition 1.1. A function $f\colon G \to \mathbb{C}$ is a *class function* or *central* if, for all $g, x \in G$, $f(xgx^{-1}) = f(g) \iff$ for all $g, x \in G$, $f(xg) = f(gx)$. Equivalently, f is constant on conjugacy classes of G, i.e. $y \in C(x) \implies f(y) = f(x)$. We define $Z \subseteq L^2(G) = \mathbb{C}[G]$ to be the vector space of all class functions.

Note that the positive definite Hermitian inner product $\langle \cdot, \cdot \rangle$ on $L^2(G)$ defines a positive definite Hermitian inner product $\langle \cdot, \cdot \rangle$ on Z by restriction.

Example 1.2. 1) If V is a G-representation and χ_V is its character, then χ_V is a class function.
2) Let $x \in G$ and let $C(x)$ be the conjugacy class of x. Define the *characteristic function* $f_{C(x)}$ as follows:

$$f_{C(x)}(g) = \begin{cases} 1, & \text{if } g \in C(x); \\ 0, & \text{if } g \notin C(x). \end{cases}$$

Then $f_{C(x)}$ is a class function and the set of $f_{C(x)}$ is clearly a basis for Z. It is an orthogonal basis of Z with respect to the Hermitian inner product, i.e. $\langle f_{C(x)}, f_{C(y)} \rangle = 0$ if $C(x) \neq C(y)$, but it is not unitary as

$$\langle f_{C(x)}, f_{C(x)} \rangle = \frac{\#(C(x))}{\#(G)}.$$

Finally, it is clear that $\dim Z$ is equal to the number of conjugacy classes of G, since the $f_{C(x)}$ are a basis for Z.

Let V be a G-representation and let $f\colon G \to \mathbb{C}$ be a function. Define a linear map $F_{V,f}\colon V \to V$ by:

$$F_{V,f} = \sum_{g \in G} f(g)\rho_V(g).$$

Clearly, if $V \cong V_1 \oplus V_2$, then $F_{V_1 \oplus V_2, f} = F_{V_1,f} \oplus F_{V_2,f}$.

Proposition 1.3. *Let* $f\colon G \to \mathbb{C}$ *be a class function, and let* V *be an irreducible* G*-representation. If* $F_{V,f}$ *is defined as above, then* $F_{V,f} = t\,\mathrm{Id}$*, where*

$$t = \frac{\#(G)\langle f, \overline{\chi_V}\rangle}{\dim V}.$$

Proof. First we claim that, for a class function f, $F_{V,f}$ is a G-morphism (for every G-representation, not necessarily irreducible). We must show that

$$\rho_V(h) \circ F_{V,f} \circ \rho_V(h)^{-1} = F_{V,f}.$$

Using the definition of $F_{V,f}$,

$$\begin{aligned}\rho_V(h) \circ F_{V,f} \circ \rho_V(h)^{-1} &= \sum_{g\in G} f(g)\rho_V(h) \circ \rho_V(g) \circ \rho_V(h)^{-1} \\ &= \sum_{g\in G} f(g)\rho_V(hgh^{-1}) \\ &= \sum_{g\in G} f(hgh^{-1})\rho_V(hgh^{-1}) \\ &= \sum_{g\in G} f(g)\rho_V(g) = F_{V,f},\end{aligned}$$

where we have used the fact that f is a class function to conclude that $f(g) = f(hgh^{-1})$, and also the fact that, for a fixed $h \in G$, the elements hgh^{-1} run through all elements of G.

Thus $F_{V,f}$ is a G-morphism. By Schur's lemma, if V is irreducible, then $F_{V,f} = t\,\mathrm{Id}$ for some $t \in \mathbb{C}$. Taking traces, we find that

$$\mathrm{Tr}\, F_{V,f} = t(\dim V).$$

On the other hand, by definition,

$$\mathrm{Tr}\, F_{V,f} = \sum_{g\in G} f(g)\chi_V(g) = \#(G)\langle f, \overline{\chi_V}\rangle.$$

Equating these gives the formula for t. ∎

Proposition 1.4. (i) *If* f *is a class function and* $\langle f, \chi_V\rangle = 0$ *for all irreducible representations* V*, then* $f = 0$*.*

(ii) *If* $V_1, \dots, V_h$ *are the irreducible representations of* G*, in the sense that* $V_1, \dots, V_h$ *are irreducible representations such that (1) For* $i \neq j$*,* V_i *is not isomorphic to* V_j *and (2) Every irreducible* G*-representation is isomorphic to* V_i *for some* i*, then the characters* $\chi_{V_1}, \dots, \chi_{V_h}$ *are a unitary basis for* Z*, the vector space of class functions.*

Proof. (i) If V is an irreducible representation of G, then V^* is irreducible as well, by a HW problem. Thus, since $\chi_{V^*} = \overline{\chi_V}$, the hypothesis of (i) implies that $\langle f, \overline{\chi_V}\rangle = 0$ for every irreducible representation V of G. By Proposition 1.3,

$$F_{V,f} = \sum_{g \in G} f(g)\rho_V(g) = 0.$$

Since every representation is a direct sum of irreducible representations, it follows that $\sum_{g\in G} f(g)\rho_V(g) = 0$ for every representation V. In particular, taking $V = \mathbb{C}[G]$, it follows that

$$F_{\mathbb{C}[G],f} = \sum_{g \in G} f(g)\rho_{\mathbb{C}[G]}(g) = 0.$$

Let $1 = 1 \cdot 1$ be the identity element of the ring $\mathbb{C}[G]$ (the coefficient of $1 \in G$ is 1, and the coefficient of $g \neq 1$ is 0). Then $F_{\mathbb{C}[G],f}(1) = 0$. But $\rho_{\mathbb{C}[G]}(g)(1) = g \cdot 1 = g$, so

$$\sum_{g \in G} f(g)\rho_{\mathbb{C}[G]}(g)(1) = \sum_{g \in G} f(g) \cdot g = 0.$$

It follows that $f(g) = 0$ for all g, i.e. $f = 0$.

(ii) Since $\langle \chi_{V_i}, \chi_{V_j}\rangle = 0$ if $i \neq j$ and $= 1$ for $i = j$, the functions $\chi_{V_i}, \dots, \chi_{V_h}$ are a linearly independent subset of Z. To see that they are basis, it suffices to show that they span Z. Equivalently, it suffices to show that $\{\chi_{V_i}, \dots, \chi_{V_h}\}^\perp = \{0\}$. But this follows from (1). □

Corollary 1.5. *The number of irreducible representations of G up to isomorphism as above is equal to the number of conjugacy classes of G.*

Proof. By (ii) of the above proposition, the number of irreducible representations of G up to isomorphism is equal to $\dim Z$. On the other hand, $\dim Z$ is equal to the number of conjugacy classes of G, and equating these two expressions for $\dim Z$ gives the proof of the lemma. □

Corollary 1.6. *The group G is abelian $\iff$ every irreducible representation of G has dimension one.*

Proof. We have seen that, if G is abelian, then every irreducible representation of G has dimension one. Conversely, suppose that every irreducible representation of G has dimension one, and let h denote as usual the number of such up to isomorphism. Since $\sum_{i=1}^h d_i^2 = \#(G)$, It follows that $h = \#(G)$. Since h is also the number of conjugacy classes of G, this number is also $\#(G)$. Clearly, this is only possible if every conjugacy class has exactly one element. But this implies that G is abelian. □

We also have the following orthogonality relations:

Proposition 1.7. *With $V_1, \dots, V_h$ as above and $\chi_{V_1}, \dots, \chi_{V_h}$ the corresponding characters, then, for all $x \in G$,*

$$\boxed{\sum_{i=1}^{h} |\chi_{V_i}(x)|^2 = \frac{\#(G)}{\#(C(x))}}$$

whereas for all $y \in G$, if $y \notin C(x)$, then

$$\boxed{\sum_{i=1}^{h} \chi_{V_i}(x)\overline{\chi_{V_i}(y)} = 0}$$

Proof. Let $C(x)$ be a conjugacy class in G and let $f_{C(x)}$ be the characteristic function of $C(x)$. Since $\chi_{V_1}, \dots, \chi_{V_h}$ is a basis for the space of class functions, there exist $t_i \in \mathbb{C}$ such that

$$f_{C(x)} = \sum_{i=1}^{h} t_i \chi_{V_i}.$$

Taking inner products, and using the orthogonality relations, we find that

$$t_i = \left\langle \sum_{j=1}^{h} t_j \chi_{V_j}, \chi_{V_i} \right\rangle = \langle f_{C(x)}, \chi_{V_i} \rangle = \frac{1}{\#(G)} \sum_{g \in G} f_{C(x)}(g)\overline{\chi_{V_i}(g)}.$$

But $f_{C(x)}(g) = 0$ if g is not conjugate to x and $= 1$ if g is conjugate to x, so the last sum above is a sum of $\overline{\chi_{V_i}(g)}$ for all g conjugate to x. For such an x, $\overline{\chi_{V_i}(g)} = \overline{\chi_{V_i}(x)}$ since χ_{V_i} is a class function, and the total number of possible g is $\#(C(x))$. Thus $t_i = \dfrac{\#(C(x))}{\#(G)}\overline{\chi_{V_i}(x)}$. Hence

$$f_{C(x)} = \sum_{i=1}^{h} \frac{\#(C(x))}{\#(G)} \overline{\chi_{V_i}(x)} \chi_{V_i}.$$

Plugging in x, we see that

$$1 = f_{C(x)}(x) = \sum_{i=1}^{h} \frac{\#(C(x))}{\#(G)} \overline{\chi_{V_i}(x)} \chi_{V_i}(x) = \frac{\#(C(x))}{\#(G)} \sum_{i=1}^{h} |\chi_{V_i}(x)|^2,$$

which gives the first formula above. For the second, plug in a $y \notin C(x)$ to get

$$0 = f_{C(x)}(y) = \sum_{i=1}^{h} \frac{\#(C(x))}{\#(G)} \overline{\chi_{V_i}(x)} \chi_{V_i}(y),$$

and hence $\sum_{i=1}^{h} \overline{\chi_{V_i}(x)} \chi_{V_i}(y) = 0$. Taking complex conjugates gives

$$\sum_{i=1}^{h} \chi_{V_i}(x) \overline{\chi_{V_i}(y)} = 0$$

as well. □

2 Character tables

Given a group G, its *character table* is an $h \times h$ matrix (or table), where we plot the conjugacy classes $C(x_1)$, ..., $C(x_h)$ of G horizontally, typically starting with $C(1) = \{1\}$, and the distinct irreducible representations $V_1, \dots, V_h$ of G (up to isomorphism) vertically, typically starting with the trivial representation, and the corresponding entry in the table is the common value of χ_{V_i} on any element of $C(x_j)$. For example, the character table of S_3 is given as follows:

	1	(i,j)	(i,j,k)
1	1	1	1
ε	1	-1	1
χ_{W_2}	2	0	-1

Here, we have symbolically denoted the conjugacy class of all 2-cycles by (i,j), and similarly for 3-cycles. As for the list of characters of irreducible representations, the trivial representation $\mathbb{C}(1)$ has character the trivial homomorphism, or constant function 1, and the other dimension one representation $\mathbb{C}(\varepsilon)$ has character ε, where $\varepsilon\colon S_3 \to \{\pm 1\}$ is the sign homomorphism. The orthogonality relations imply that the columns of the table are orthogonal, viewed as vectors in $\mathbb{C}^3$ under the Hermitian inner product, and the sums of the absolute values squared as we go down a column are equal to $6/\#(C(x))$, where $C(x)$ is the corresponding conjugacy class, hence (reading from left to right) $6, 2, 3$ respectively.