More on induced representations

1 The case of a normal subgroup

Let G be a finite group and let H be a **normal** subgroup of G. For an H-representation, we want to give a formula for $\operatorname{Res}_H^G \operatorname{Ind}_H^G W$. First, some notation: if $x \in G$ and $h \in H$, then $hx = xh'$ for some $h' \in H$, where $h' = x^{-1}hx$. In particular, writing as usual $x_1 = 1, \dots, x_k$ for a set of representatives for the left cosets of H,

$$hx_i = x_i h_i(h) = x_i(x_i^{-1}hx_i).$$

This says that

$$\rho_{\operatorname{Ind}_h^G W}(h)(F_{i,w}) = F_{i,\rho_W(x_i^{-1}hx_i)}.$$

In particular, the vector subspaces $W^{(i)} = \{F_{iw} : w \in W$ are invariant under the restriction of $\rho_{\operatorname{Ind}_h^G W}$ to elements of H, i.e. they are $\rho_{\operatorname{Res}_H^G W}$-invariant subspaces.

Given $x \in G$, since H is normal, we have $i_x(H) \subseteq H$, and in fact $i_x \colon H \to H$ is an isomorphism from H to H, where by definition

$$i_x(g) = xgx^{-1}.$$

Define W_x to be the H-representation given by the homomorphism $\rho_W \circ i_x^{-1} \colon H \to \operatorname{Aut} W$. Explicitly:

$$\rho_{W_x}(g) = \rho_W(x^{-1}gx).$$

In particular, for $1 \leq i \leq k$, we have the H-representation W_{x_i}. Then the calculations above show:

Proposition 1.1. *As H-representations,*

$$\operatorname{Res}_H^G \operatorname{Ind}_H^G W \cong \bigoplus_{i=1}^{k} W_{x_i}. \quad \square$$

This formula allows us to describe when $\operatorname{Ind}_H^G W$ is irreducible. Note that, if W is reducible, say $W \cong W_1 \oplus W_2$ as H-representations, then it is easy to see that $\operatorname{Ind}_H^G W \cong \operatorname{Ind}_H^G W_1 \oplus \operatorname{Ind}_H^G W_2$, and hence is also reducible. Thus we may as well assume that W is irreducible.

Theorem 1.2. *Suppose that H is a normal subgroup of G and that W is an irreducible$H$-representation. Then $\operatorname{Ind}_H^G W$ is an irreducible G representation $\iff$ for all $x \in G$ with $x \notin H$, W_x is not H-isomorphic to W.*

Proof. Since W is irreducible, $\langle \chi_W, \chi_W \rangle_H = 1$. We wish to see when $\langle \chi_{\operatorname{Ind}_H^G W}, \chi_{\operatorname{Ind}_H^G W} \rangle_G = 1$. In any case, by Frobenius reciprocity,

$$\langle \chi_{\operatorname{Ind}_H^G W}, \chi_{\operatorname{Ind}_H^G W} \rangle_G = \langle \chi_W, \chi_{\operatorname{Res}_H^G \operatorname{Ind}_H^G W} \rangle_H = \sum_{i=1}^{k} \langle \chi_W, \chi_{W_{x_i}} \rangle_H,$$

by Proposition 1.1. For $i = 1$, $W_{x_1} = W_1 = W$ and hence $\langle \chi_W, \chi_{W_1} \rangle_H = 1$. For $i > 1$, W_{x_i} is an irreducible representation and so $\langle \chi_W, \chi_{W_{x_i}} \rangle_H = 1$ if $W_{x_i} \cong W$ and $\langle \chi_W, \chi_{W_{x_i}} \rangle_H = 0$ if W_{x_i} is not H-isomorphic to W. Thus $\operatorname{Ind}_H^G W$ is irreducible $\iff$ $\langle \chi_{\operatorname{Ind}_H^G W}, \chi_{\operatorname{Ind}_H^G W} \rangle_G = 1$ $\iff$ for all $i > 1$, W_{x_i} is not H-isomorphic to W.

It remains to show that the statement that, for all $i > 1$, W_{x_i} is not H-isomorphic to W, is equivalent to the statement that, for all $x \notin H$, W_x is not H-isomorphic to W. Clearly, since for $i > 1$ $x_i \notin H$, the second statement implies the first. Conversely, suppose the first statement. Let $x \in G$, $x \notin H$. Then x is in some left coset $x_i H$, and the assumption $x \notin H$ is equivalent to saying that $i > 1$. Thus we can write $x = x_i h$ for some $i > 1$. It follows that

$$\begin{aligned} \rho_W \circ i_x^{-1} &= \rho_W \circ i_{(x_i h)^{-1}} = \rho_W \circ i_h^{-1} \circ i_{x_i}^{-1} \\ &= \rho_W(h)^{-1} \circ (\rho_W \circ i_{x_i}^{-1}) \circ \rho_W(h). \end{aligned}$$

It follows that the representations W_x and W_{x_i} are conjugate by some element in $\operatorname{Aut} W$, namely $\rho_W(h)^{-1}$. Hence W_x and W_{x_i} are H-isomorphic. Thus, if W_{x_i} is not H-isomorphic to W for all $i > 1$, then W_x is not H-isomorphic to W for all $x \notin H$. □

Example 1.3. (1) If $W = \mathbb{C}$ is the trivial representation and $H \neq G$, then W_x is isomorphic to W for every $x \in G$, hence $\operatorname{Ind}_H^G \mathbb{C}$ is not irreducible. In fact, we know that $\operatorname{Ind}_H^G \mathbb{C} \cong \mathbb{C}[G/H]$ always contains a subspace isomorphic to the trivial representation of G, and hence is not irreducible if $\dim \mathbb{C}[G/H] = (G : H) > 1$, i.e. if $H \neq G$. (If $H = G$, then the condition that W_x is not H-isomorphic to W for all $x \notin H$ is vacuously satisfied, and in fact $\operatorname{Ind}_G^G \mathbb{C} \cong \mathbb{C}$ is trivial but irreducible.)

(2) Suppose that $G = D_n$ and $H = \langle \alpha \rangle$. Then we can take $x_2 = \tau$ and $i_\tau^{-1}(\alpha^k) = i_\tau(\alpha^k) = \alpha^{-k}$. Thus, for $W = W_a = \mathbb{C}(\lambda_a)$, the 1-dimensional representation corresponding to the homomorphism $\lambda_a \colon H \to \mathbb{C}^*$ defined by $\lambda_a(\alpha^k) = e^{2\pi i a k/n}$, we have

$$(W_a)_{x_2} = W_{-a}.$$

Note that a is naturally an element of $\mathbb{Z}/n\mathbb{Z}$, since $W_a \cong W_b \iff a \equiv b \pmod{n}$. The condition that, for all $x \in H$, $(W_a)_x$ is not isomorphic to W_a is then the condition that $-a$ and a are not congruent mod n, i.e. that $2a \not\equiv 0 \pmod{n}$. Note that $2a \equiv 0 \pmod{n} \iff a = 0$ or n is even, say $n = 2m$, and $a \equiv m \pmod{n}$. In conclusion, we see that $\operatorname{Ind}_H^{D_n} W_a$ is irreducible unless $a = 0$ or $n = 2m$, and $a \equiv m \pmod{n}$. Of course, we could also verify this by a direct computation.

For the remainder of this section, we specialize still further, to the case where H is a subgroup of G of index 2. Of course, H is known to be normal in this case. An interesting example to keep in mind is $G = S_n$, $H = A_n$. In general, G/H is a group of order 2, and there is a homomorphism $\varepsilon \colon G \to \mathbb{C}^*$ defined by $\varepsilon(g) = 1$ if $h \in H$ and $\varepsilon(g) = -1$ if $g \notin H$.In case $G = S_n$, $H = A_n$, then ε is the sign homomorphism. We also fix an element $x \in G - H$ and have the resulting isomorphism $i_x^{-1} \colon H \to H$. Recall that, if W is an H-representation corresponding to $\rho_W \colon H \to \operatorname{Aut} W$, then we have defined the H-representation W_x which corresponds to the homomorphism $\rho_W \circ i_x^{-1}$. It is in fact independent of the choice of x up to H-isomorphism.

Our main interest is the following question: given an irreducible G-representation, when is $\operatorname{Res}_H^G V$ still irreducible? The answer is given by the following:

Theorem 1.4. *Let G be a finite group and let H be a subgroup of G of index 2. Let V be an irreducible G-representation and let $W = \operatorname{Res}_H^G V$. Finally, let $V \otimes \varepsilon$ be the representation corresponding to the homomorphism $\rho_{V\otimes\varepsilon} = \varepsilon\rho_V$. Then exactly one of the following holds:*

(i) *V is G-isomorphic to $V \otimes \varepsilon$, W is H-isomorphic to W_x, and W is H-isomorphic to $W' \oplus W'_x$, where W' and hence W'_x are irreducible representations with W' not H-isomorphic to W'_x. Finally,* $\dim V$ *is even and*
$$V \cong \operatorname{Ind}_H^G W' \cong \operatorname{Ind}_H^G W'_x.$$

(ii) *V is not G-isomorphic to $V \otimes \varepsilon$, W is irreducible, W is H-isomorphic to W_x, and*

$$\operatorname{Ind}_H^G W \cong V \oplus (V \otimes \varepsilon).$$

Finally, every irreducible H-representation arises this way, either as an irreducible summand of $\operatorname{Res}_H^G V$ where V is an irreducible G-representation G-isomorphic to $V \otimes \varepsilon$, or as $\operatorname{Res}_H^G V$ where V is an irreducible G-representation which is not G-isomorphic to $V \otimes \varepsilon$.

Proof. As a general remark, if H is normal, then, for all $x \in G$, $(\operatorname{Res}_H^G V)_x \cong \operatorname{Res}_H^G V$: For $x \in G$, let V_x be the G-representation defined by $\rho_V \circ i_x^{-1}$. Then V_x is is G-isomorphic to V since ρ_V and $\rho_V \circ i_x^{-1}$ differ by conjugation by $\rho_V(x)^{-1}$. Then $\operatorname{Res}_H^G(V_x) \cong \operatorname{Res}_H^G V$, but clearly $\operatorname{Res}_H^G(V_x) = (\operatorname{Res}_H^G V)_x$. Thus, in both (i) and (ii) above, W is H-isomorphic to W_x.

Note also that $\chi_{V\otimes\varepsilon} = \varepsilon\chi_V$, and thus

$$\chi_{V\otimes\varepsilon}(g) = \begin{cases} \chi_V(g), & \text{if } g \in H; \\ -\chi_V(g), & \text{if } g \notin H. \end{cases}$$

Thus V is G-isomorphic to $V \otimes \varepsilon \iff \chi_V = \chi_{V\otimes\varepsilon} \iff \chi(g) = -\chi_V(g)$ for all $g \notin H \iff \chi(g) = 0$ for all $g \notin H$.

Since V is irreducible,

$$\langle \chi_V, \chi_V \rangle_G = \frac{1}{\#(G)} \sum_{g\in G} |\chi_V(g)|^2 = 1.$$

Hence $\sum_{g\in G} |\chi_V(g)|^2 = \#(G) = 2\#(H)$. We rewrite this as

$$\begin{aligned} 2\#(H) &= \sum_{g\in G} |\chi_V(g)|^2 = \sum_{h\in H} |\chi_V(h)|^2 + \sum_{g\notin H} |\chi_V(g)|^2 \\ &= \#(H)\langle \chi_W, \chi_W \rangle_H + \sum_{g\notin H} |\chi_V(g)|^2. \end{aligned}$$

Now $\langle \chi_W, \chi_W \rangle_H$ is a positive integer n and $\#(H)\langle \chi_W, \chi_W \rangle_H = n\#(H)$. Also, since $|\chi_V(g)|^2 \geq 0$, we see that

$$n\#(H) \leq 2\#(H),$$

hence $n \leq 2$ with equality $\iff$ $\chi_V(g) = 0$ for all $g \notin H$ $\iff$ V is G-isomorphic to $V \otimes \varepsilon$.

Case I: $n = 2$. As noted above, this case happens $\iff$ V is G-isomorphic to $V \otimes \varepsilon$. If $W = \operatorname{Res}_H^G$ is a direct sum of representations $U_i^{m_i}$, $1 \leq i \leq r$, where the U_i are pairwise non-isomorphic, then $\sum_{i=1}^r m_i^2 = 2$. The only way this can happen is that $r = 2$ and $m_1 = m_2 = 1$, i.e. $W \cong W' \oplus W''$, where W' and W'' are irreducible and W' is not isomorphic to W''. Let $d = \dim V$, so that $d = \dim W' + \dim W''$. Consider $\operatorname{Ind}_H^G W'$. By Frobenius reciprocity,

$$\langle \chi_V, \chi_{\operatorname{Ind}_H^G W'} \rangle_G = \langle \chi_W, \chi_{W'} \rangle_H = \langle \chi_{W'} + \chi_{W''}, \chi_{W'} \rangle_H = 1,$$

since W' and W'' are irreducible but not isomorphic. In particular, V is a direct summand of $\operatorname{Ind}_H^G W'$, and hence $\dim V = d \leq \dim \operatorname{Ind}_H^G W'$. By symmetry, V is a direct summand of $\operatorname{Ind}_H^G W''$, and hence $\dim V = d \leq \dim \operatorname{Ind}_H^G W''$. Adding, we see that

$$2d \leq \dim \operatorname{Ind}_H^G W' + \dim \operatorname{Ind}_H^G W'' = 2 \dim W' + 2 \dim W'' = 2d.$$

The only way that this can hold is for $\dim V = \dim \operatorname{Ind}_H^G W' = \dim \operatorname{Ind}_H^G W''$, but then $V \cong \operatorname{Ind}_H^G W'$ and $V \cong \operatorname{Ind}_H^G W''$ since V is isomorphic to a summand of $\operatorname{Ind}_H^G W'$ with the same dimension as $\operatorname{Ind}_H^G W'$, and similarly for $\operatorname{Ind}_H^G W''$. Since $V \cong \dim \operatorname{Ind}_H^G W'$,

$$W = \operatorname{Res}_H^G V \cong \operatorname{Res}_H^G \operatorname{Ind}_H^V W' \cong W' \oplus W'_x,$$

but also $W \cong W' \oplus W''$, where W' and W' are non-isomorphic. It follows that $W'' \cong W'_x$. Finally, $\dim V = 2 \dim W'$ and hence $\dim V$ is even.

Case II: $n < 2$, hence $n = 1$. In this case, V and $V \otimes \varepsilon$ are not isomorphic. Moreover

$$\operatorname{Ind}_H^G W = \operatorname{Ind}_H^G \operatorname{Res}_H^G V = V \otimes \mathbb{C}[G/H].$$

By definition $\mathbb{C}[G/H]$ is a vector space of dimension 2 with basis $e_1 = H$ and $e_2 = xH$ for any $x \notin H$. Moreover, $\rho_{\mathbb{C}[G/H]}(g)(e_1) = e_1$ and $\rho_{\mathbb{C}[G/H]}(g)(e_2) = e_2$ if $g \in H$ and $\rho_{\mathbb{C}[G/H]}(g)(e_1) = e_2$ and $\rho_{\mathbb{C}[G/H]}(g)(e_2) = e_1 2$ if $g \notin H$. It follows that $e_1 + e_2$ is a G-invariant vector, and hence spans a subspace G-isomorphic to the trivial representation $\mathbb{C} = \mathbb{C}(1)$. Also $e_1 - e_2 = v$ satisfies $\rho_{\mathbb{C}[G/H]}(g) = \varepsilon(g)v$, hence v spans a subspace G-isomorphic to the representation $\mathbb{C}(\varepsilon)$. Thus

$$\operatorname{Ind}_H^G W \cong V \oplus (V \otimes \varepsilon).$$

In particular, by Theorem 1.2, $W \cong W_x$.

Finally, we must show that every irreducible representation of H arises in this way. We leave this as an exercise. $\square$

Example 1.5. (1) For $G = D_n$ and $H = \langle \alpha \rangle$, we have seen that every irreducible representation of D_n has dimension 1 or 2. If V is an irreducible 2-dimensional representation of D_n, then $\mathrm{Res}_H^{D_n} V$ is never irreducible since H is abelian. Thus $\mathrm{Res}_H^G V = W' \oplus W'_\tau$. Every irreducible representation of H is of the form W_a for some $a \in \mathbb{Z}/n\mathbb{Z}$, where W_a corresponds to the homomorphism λ_a as in Example 1.3(2). Then $(W_a)_\tau = W_{-a}$, where $2a \not\equiv 0 \pmod{n}$. Moreover, in this case $V \cong \mathrm{Ind}_H^{D_n} W_a \cong \mathrm{Ind}_H^{D_n} W_{-a}$.

(2) Let $G = S_4$ and $H = A_4$. We have seen that the standard permutation representation of S_4 on $\mathbb{C}^4$ has a direct sum decomposition as $\mathbb{C}^4 \cong V_3 \oplus \mathbb{C}$, where V_3 is irreducible. The representation $V_3 \otimes \varepsilon$ is not isomorphic to V_3. There are the two 1-dimensional representations $\mathbb{C}$ and $\mathbb{C}(\varepsilon)$. Finally, there is a 2-dimensional representation V_2, unique up to isomorphism. It comes from the homomorphism $S_4 \to S_4/H \cong S_3$ by taking the 2-dimensional irreducible representation of S_3. Note that

$$1^2 + 1^2 + 2^2 + 3^2 + 3^2 = 24 = \#(S_4),$$

so these are all the irreducible representations of S_4 up to isomorphism.

As for A_4, the quotient homomorphism $A_4 \to A_4/H \cong \mathbb{Z}/3\mathbb{Z}$ gives three 1 dimensional representations, the trivial representation $\mathbb{C}$ and two others $\mathbb{C}(\lambda_1)$ and $\mathbb{C}(\lambda_2)$. Finally, the representation V_3 of S_4 remains irreducible when restricted to A_4, which we saw directly or by (2) of Theorem 1.4 above. (Note also that, as $\dim V_3$ is odd, we must be in Case (2).) Let $W_3 = \mathrm{Res}_{A_4}^{S_4} V_3$. As

$$1^2 + 1^2 + 1^2 + 3^2 = 12 = \#(A_4),$$

we have found all the irreducible representations of A_4 up to isomorphism.

We have already noted that V_3 satisfies case (2) of Theorem 1.4, and hence so does $V_3 \otimes \varepsilon$; in fact, with G and H as in the theorem, we always have $\mathrm{Res}_H^G V = \mathrm{Res}_H^G (V \otimes \varepsilon)$. As for V_2, it must satisfy $V_2 \otimes \varepsilon \cong V_2$ since there is a unique 2-dimensional representation up to isomorphism. Of course, there are many ways of checking this directly. Hence we are in case (1) and $\mathrm{Res}_{A_4}^{S_4} V_2 \cong W' \oplus W'_x$, where W' and W'_x are 1-dimensional and W' and W'_x are not isomorphic. Thus neither W' nor W'_x are trivial, and hence (possibly after relabeling) $W' \cong \mathbb{C}(\lambda_1)$ and $W'_x \cong \mathbb{C}(\lambda_2)$. Thus $\mathrm{Res}_{A_4}^{S_4} V_2 \cong \mathbb{C}(\lambda_1) \oplus \mathbb{C}(\lambda_2)$ and $V_2 \cong \mathrm{Ind}_{A_4}^{S_4} \mathbb{C}(\lambda_1) \cong \mathrm{Ind}_{A_4}^{S_4} \mathbb{C}(\lambda_2)$.

2 Mackey's theorems

Mackey proved two theorems about induced representations. The first describes $\operatorname{Res}_H^G \operatorname{Ind}_H^G W$ for an arbitrary, not necessarily normal subgroup H of G and an H-representation W. With essentially the same amount of effort, the theorem describes $\operatorname{Res}_K^G \operatorname{Ind}_H^G W$ where K is another subgroup of G, possibly equal to H. Using this, the second theorem gives a necessary and sufficient condition for $\operatorname{Ind}_H^G V$ to be irreducible. Both theorems use the concept of a double coset, which we now define:

Definition 2.1. Let G be a group, let $x \in G$, and let H and K be two subgroups of G. A *double coset* KxH of G is a subset of the form

$$KxH = \{kxh : k \in K, h \in H\}.$$

Thus a left coset for H is a double coset $\{1\}xH$ and a right coset is a double coset $Hx\{1\}$. Just as a left coset for H is an equivalence class for the equivalence relation $x_1 \sim x_2 \iff x_1 = x_2h$ for some $h \in H$, a double coset KxH is an equivalence class for the equivalence relation $x_1 \sim x_2 \iff$ there exist $h \in H$ and $k \in K$ such that $x_1 = kx_2h$. (This is easily checked to be an equivalence relation.) In particular, given H and K, G is a disjoint union of double cosets and (if G is finite) there exists a set of representatives $y_1, \dots, y_n \in G$ such that every element of G is in exactly one double coset Ky_iH. In other words, for every $g \in G$, there exists a unique i, $1 \le i \le n$, and unique elements $h \in H$ and $k \in K$ such that $g = ky_ih$. However, unlike the case of left or right cosets, the number of elements of a double coset does not have to divide the order of G, and in particular different double cosets can have different numbers of elements. We denote the set of double cosets (for K and H) by $K\backslash G/H$.

Finally, note that every double coset KxH is a union of left cosets of H (and also a union of right cosets of K).

We now state Mackey's first theorem. For a finite group H and two subgroups H and K of H, we fix a set of representatives $y_1, \dots, y_n$ for the double cosets as above. Define a subgroup H_i of K via

$$H_i = y_iHy_i^{-1} \cap K \le K.$$

If W is an H-representation corresponding to $\rho_W \colon H \to \operatorname{Aut} W$, define a representation W_i of H_i by

$$\rho_{W_i} = \operatorname{Res}_{H_i}^{y_iHy_i^{-1}} \rho_W \circ i_{y_i}^{-1}.$$

Here $i_{y_i}^{-1}$ is an isomorphism from $y_iHy_i^{-1}$ to H, thus $\rho_W \circ i_{y_i}^{-1}$ defines a representation of $y_iHy_i^{-1}$. Explicitly, every element of $y_iHy_i^{-1}$ is equal to $y_ihy_i^{-1}$ for a unique $h \in H$, and then by definition

$$\rho_W \circ i_{y_i}^{-1}(y_ihy_i^{-1}) = \rho_W(h).$$

We can then restrict $\rho_W \circ i_{y_i}^{-1}$ to the subgroup H_i of $y_iHy_i^{-1}$, and in this way we obtain W_i. Note that, if H is normal and $K = H$, then $y_iHy_i^{-1} = H$, $H_i = y_iHy_i^{-1} \cap H = H$, and $W_i = W_{y_i}$ as previously defined.

Theorem 2.2 (Mackey)**.** *In the above notation,*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W = \bigoplus_{i=1}^{n} \operatorname{Ind}_{H_i}^K W_i.$$

Proof. We start with a general group theory lemma:

Lemma 2.3. *Let H_1 and H_2 be two subgroups of G and define*

$$H_1H_2 = \{h_1h_2 : h_1 \in H_1, h_2 \in H_2\},$$

so that H_1H_2 is a union of left cosets (but it is not in general a subgroup of G unless one of H, K is normal). We define H_1H_2/H_2 to be the set of left cosets of H_2 of the form xH_2 for $x \in H_1H_2$. Then the function $\tilde{f}\colon H_1 \to H_1H_2/H_2$ defined by $\tilde{f}(h) = hH_2$ induces a bijection

$$f\colon H_1/H_1 \cap H_2 \to H_1H_2/H_2.$$

Proof. It is straightforward to check that f is surjective and that $f(h) = f(h') \iff h = h'h''$ for some $h'' \in H_1 \cap H_2$. □

Returning to the proof of Mackey's theorem, since Ky_iH is a disjoint union of left cosets of H, we can write

$$Ky_iH = \bigcup_{j=1}^{k_i} x_{ij}H,$$

where the $x_{ij} \in G$, $1 \leq i \leq n, 1 \leq j \leq k_i$ are a set of representatives for the left cosets of H. Then we can write

$$Ky_iHy_i^{-1} = \bigcup_{j=1}^{k_i} x_{ij}y_i^{-1}y_iHy_i^{-1},$$

a disjoint union of cosets $(x_{ij}y_i^{-1})y_iHy_i^{-1}$ for the subgroup $y_iHy_i^{-1}$. Also, if $z_1, \dots, z_{k_i}$ are any set of representatives for $Ky_iHy_i^{-1}/y_iHy_i^{-1}$, then $Ky_iHy_i^{-1}$ is a disjoint union $\bigcup_{j=1}^{k_i} z_iy_iHy_i^{-1}$ and then it follows that $Ky_iH = \bigcup_{j=1}^{k_i} z_jy_iH$. In other words, we can choose the x_{ij} to be of the form z_jy_i for any set of representatives $z_1, \dots z_{k_i}$ of $Ky_iHy_i^{-1}/y_iHy_i^{-1}$.

Applying Lemma 2.3 to the case where $H_1 = K$ and $H_2 = y_iHy_i^{-1}$: we can choose a set of representatives $z_1, \dots z_{k_i}$ for $Ky_iHy_i^{-1}/y_iHy_i^{-1}$ of the form z_j, where the $z_j \in K$ are a set of representatives for $K/y_iHy_i^{-1} \cap K = K/H_i$. Thus, taking $x_{ij} = z_jy_i$ and hence $z_j = x_{ij}y_i^{-1}$, we can assume that $x_{ij}y_i^{-1} \in K$ and that the $x_{ij}y_i^{-1}$, $1 \leq j \leq k_i$, are a set of representatives for the left cosets K/H_i.

Now let $V = \operatorname{Ind}_H^G W$. Then we have seen that $V \cong \bigoplus_{r=1}^k W^{(r)}$, where $k = (G : H)$ and the subspaces $W^{(r)}$ are indexed by a set of representatives for G/H. In our case, we have the set of representatives x_{ij} indexed by i and j, and so can write the direct sum as follows:

$$V \cong \bigoplus_{i,j} W^{(i,j)} = \bigoplus_{i=1}^{n} \left(\bigoplus_{j=1}^{k_i} W^{(i,j)} \right),$$

where

$$W^{(i,j)} = \{F \in \operatorname{Ind}_H^G W : F(g) = 0 \text{ if } g \notin x_{ij}H\}.$$

Moreover, $W^{(i,j)}$ is spanned by functions $F_{i,j,w}$, where $\rho_{\operatorname{Ind}_H^G W}(g)$ acts on $F_{i,j,w}$ as follows: if $gx_{ij} = x_{k\ell}h_{ij}(g)$, then

$$\rho_{\operatorname{Ind}_H^G W}(g)(F_{i,j,w}) = F_{k,\ell,\rho_W(h_{ij}(g))(w)}.$$

So it suffices to show that the subspaces $\bigoplus_{j=1}^{k_i} W^{(i,j)}$ are K-invariant and that each such subspace is K-isomorphic to $\operatorname{Ind}_{H_i}^K W_i$. To see this, note that, if $k \in K$, then $kx_{ij} \in Ky_iH$, and so $kx_{ij} = x_{i\ell}h_{ij}(k)$ for some $h_{ij}(k) \in H$ (since Ky_iH is a union of the $x_{i\ell}H$). This says that the subspaces $\bigoplus_{j=1}^{k_i} W^{(i,j)}$ are K-invariant and that

$$\rho_{\operatorname{Ind}_H^G}(k)(F_{i,j,w}) = F_{i,\ell,\rho_W(h_{ij}(k))(w)}.$$

To compare this K-representation with $\operatorname{Ind}_{H_i}^K W_i$, first note that, fixing i, as $kx_{ij} = x_{i\ell}h_{ij}(k)$ and $z_j = x_{ij}y_i^{-1}$,

$$kz_j = kx_{ij}y_i^{-1} = x_{i\ell}h_{ij}(k)y_i^{-1} = z_\ell(y_ih_{ij}(k)y_i^{-1}).$$

Moreover, since $k, z_j, z_\ell \in K$, it follows that $y_i h_{ij}(k) y_i^{-1} \in y_i H y_i^{-1} \cap K = H_i$. The above says that

$$\operatorname{Ind}_{H_i}^K W_i \cong \bigoplus_{j=1}^{k_i} W_i^{(j)},$$

where $W_i^{(j)}$ is spanned by functions which we denote by $G_{i,j,w}$ and

$$\rho_{\operatorname{Ind}_{H_i}^K}(k)(G_{i,j,w}) = G_{i,\ell,\rho_W(h_{ij}(k))(w)}.$$

Comparing, we see that, after identifying $F_{i,j,w}$ with $G_{i,j,w}$, the action of $k \in K$ on $\bigoplus_{j=1}^{k_i} W^{(i,j)}$ is the same as the action of $k \in K$ on $\operatorname{Ind}_{H_i}^K W_i$. Thus

$$\bigoplus_{j=1}^{k_i} W^{(i,j)} \cong \operatorname{Ind}_{H_i}^K W_i$$

and hence $\operatorname{Res}_K^G \operatorname{Ind}_H^G W = \bigoplus_{i=1}^n \operatorname{Ind}_{H_i}^K W_i$ as claimed. □

We turn now to Mackey's second theorem. Before stating it, we give a preliminary definition:

Definition 2.4. Let G be a finite group and let V_1 and V_2 be two G-representations. We say that V_1 and V_2 are *disjoint* if no irreducible summand of V_1 is isomorphic to an irreducible summand of V_2, or equivalently if $\langle \chi_{V_1}, \chi_{V_2} \rangle_G = 0$.

We can then state the following:

Theorem 2.5 (Mackey's irreducibility criterion)**.** *Let G be a finite group, H a subgroup of G, and W an H-representation. Then $\operatorname{Ind}_H^G W$ is irreducible $\iff$ the following two conditions hold:*

(i) *W is an irreducible H-representation.*

(ii) *For every $x \in G - H$, if we set W_x to be the representation of xHx^{-1} corresponding to $\rho_W \circ i_x^{-1}$ and $H_x = xHx^{-1} \cap H$, the representations $\operatorname{Res}_{H_x}^H W$ and $\operatorname{Res}_{H_x}^{xHx^{-1}} W_x$ are disjoint H_x-representations.*

Remark 2.6. (1) If H is normal, then $H_x = H$ and the statement is just that of Theorem 1.2.

(2) The subgroup H_x only depends on the double coset HxH up to conjugation by an element of H.

Proof. Choose a set $y_1, \dots, y_n$ for the double cosets HxH. We might as well assume that $y_1 = 1$ and thus that $Hy_1H = H1H = H$ and that $i_{y_1}^{-1} = \mathrm{Id}$. Since G is a disjoint union of the Hy_iH,

$$G - H = \bigcup_{i>1} Hy_iH.$$

Let $H_i = y_iHy_i^{-1} \cap H$, so that $H_1 = 1$, and define $W_i = \mathrm{Res}_{H_i}^{y_iHy_i^{-1}} W_{y_i}$. In particular, $W_1 \cong W$.

The representation $\mathrm{Ind}_H^G W$ is irreducible $\iff \langle \chi_{\mathrm{Ind}_H^G W}, \chi_{\mathrm{Ind}_H^G W}\rangle_G = 1$. By Frobenius reciprocity and Mackey's Theorem,

$$\begin{aligned}\langle \chi_{\mathrm{Ind}_H^G W}, \chi_{\mathrm{Ind}_H^G W}\rangle_G &= \langle \chi_W, \chi_{\mathrm{Res}_H^G \mathrm{Ind}_H^G W}\rangle_H \\ &= \sum_i \langle \chi_W, \chi_{\mathrm{Ind}_{H_i}^H W_i}\rangle_H \\ &= \sum_i \langle \chi_{\mathrm{Res}_{H_i}^H W}, \chi_{W_i}\rangle_{H_i},\end{aligned}$$

where we have used Frobenius reciprocity twice and Mackey's theorem to write $\mathrm{Res}_H^G \mathrm{Ind}_H^G W \cong \bigoplus_i \mathrm{Ind}_{H_{y_i}}^H W_i$. In the last sum above, for $i = 1$,

$$\langle \chi_{\mathrm{Res}_{H_1}^H W}, \chi_{W_1}\rangle_{H_1} = \langle \chi_W, \chi_W\rangle_H$$

is a positive integer, and it is 1 $\iff$ W is irreducible. As for the remaining terms $\langle \chi_{\mathrm{Res}_{H_i}^H W}, \chi_{W_i}\rangle_{H_i}$ for $i > 1$, they are all nonnegative integers, and they are 0 $\iff$ the representations $\mathrm{Res}_{H_i}^H W$ and $W_i = \mathrm{Res}_{H_i}^{y_iHy_i^{-1}} W_{y_i}$ are disjoint as previously defined. This is condition (ii) of the theorem for the elements $x = y_i$, $i > 1$, which are exactly the $y_i \notin H = Hy_1H$. Thus $\mathrm{Ind}_H^G W$ is irreducible $\iff$ W is irreducible and $\mathrm{Res}_{H_i}^H W$ and $W_i = \mathrm{Res}_{H_i}^{y_iHy_i^{-1}} W_{y_i}$ are disjoint for all $i > 1$. So it suffices to show that condition (ii) for all $x \notin H$ is equivalent to condition (ii) for the $y_i \notin H$. One direction is obvious: if (ii) holds for all $x \notin H$, then it holds for all $y_i \notin H$. Conversely, suppose that (ii) holds for all $y_i \notin H$. Given an arbitrary $x \notin H$, we can write $x = hy_ih'$ for some $h, h' \in H$, and $i > 1$, since G is a disjoint union of the double cosets Hy_iH. Then a straightforward argument shows that i_h^{-1} is an isomorphism from H_x to H_{y_i} which identifies $\mathrm{Res}_{H_x}^H W$ with $\mathrm{Res}_{H_i}^H W$ and $\mathrm{Res}_{H_x}^{xHx^{-1}} W_x$ with $\mathrm{Res}_{H_i}^{y_iHy_i^{-1}} W_{y_i}$. Thus $\mathrm{Res}_{H_x}^H W$ and $\mathrm{Res}_{H_x}^{xHx^{-1}} W_x$ are disjoint H_x-representations for all $x \notin H$ $\iff$ $\mathrm{Res}_{H_i}^H W$ and $W_i = \mathrm{Res}_{H_i}^{y_iHy_i^{-1}} W_{y_i}$ are disjoint H_i-representations for all $i > 1$. $\square$