Problem set 10 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Let G be a finite group. Let (V, π) and (W, ρ) be representations of G. Prove that if $Ker(\rho) \not\supset Ker(\pi)$ then (W, ρ) is not isomorphic to a direct summand of $(V^{\otimes n}, \pi^{\otimes n})$ for any $n \geq 0$. Here $\ker(\pi)$ denotes the kernel of the group homomorphism π.

Exercise 2.[1] Let G be a finite group. Let (V, π) and (W, ρ) be representations of G. Denote χ_π and χ_ρ their characters and denote $\chi_{\pi^{\otimes n}}$ the character of the nth tensor power of (V, π) (for $n = 0$ you get the character of the 1-dimensional trivial representation). Recall that for class functions f_1, f_2 on G we set

$$(f_1, f_2) = \frac{1}{|G|} \sum\nolimits_{g \in G} f_1(g) \overline{f_2(g)}$$

(1) Write the power series

$$H(t) = \sum\nolimits_{n \geq 0} (\chi_{\pi^{\otimes n}}, \chi_\rho) t^n$$

as a rational function in t. Hint: group terms belonging to a fixed $g \in G$ similar to what we did in the lectures for certain Poincaré series.

(2) From your formula in (1) conclude that $H(t)$ is nonzero if $Ker(\rho) \supset Ker(\pi)$. Hint: look at a suitable pole of $H(t)$.

(3) Conclude that if $Ker(\rho) \supset Ker(\pi)$ and (W, π) is irreducible, then (W, ρ) is isomorphic to a direct summand of $(V^{\otimes n}, \pi^{\otimes n})$ for some $n \geq 0$.

(4) Give an example where $Ker(\rho) \supset Ker(\pi)$ but (W, ρ) is not isomorphic to a direct summand of $(V^{\otimes n}, \pi^{\otimes n})$ for any $n \geq 0$.

(5) **Optional** Assume $\pi(g)$ is not a multiple of id_V except if $\pi(g) = \mathrm{id}_V$ and that $Ker(\rho) \supset Ker(\pi)$. Show (W, ρ) is isomorphic to a direct summand of $(V^{\otimes n}, \pi^{\otimes n})$ for some $n \geq 0$ Hints: Namely, let $(W_1, \rho_1), \ldots, (W_r, \rho_r)$ be its irreducible constituents. Then for each i we are going to show that for $n \gg 0$ the irreducible representation (W_i, ρ_i) occurs with high multiplicity in $(V^{\otimes n}, \pi^{\otimes n})$. Namely, this should follow from the argument with the existence of a first order pole in the rational function $H_i(T) = \sum_{n \geq 0} (\chi_{\pi^{\otimes n}}, \chi_{\rho_i}) t^n$.

Exercise 3. Let G be a finite group and let (V, π) be a faithful representation, i.e., the map $\pi : G \to GL(V)$ is injective. Prove that every irreducible representation (W, ρ) is isomorphic to a direct summand of $(V^{\otimes n}, \pi^{\otimes n})$ for some $n \geq 0$. Hint: Use the result of Exercise 2.

Exercise 4. Let G be a finite group. Suppose we have a nonempty set S of isomorphism classes of representations of G with the following properties:

(1) If (V, π) and (W, ρ) are in S, then so is $(V \otimes W, \pi \otimes \rho)$.

(2) If (V, π) is in S and (W, ρ) is isomorphic to a summand of (V, π), then (W, ρ) is in S.

Prove that there exists a surjection $G \to H$ of groups such that S consists of the isomorphism classes of those (V, π) such that $\pi : G \to GL(V)$ factors as $G \to H \to GL(V)$ for some representation $\pi' : H \to GL(V)$ of H. Hint: Use the result of Exercise 2.

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[1] Thanks to Emory for pointing out several problems with this exercise.

Exercise 5 – Optional. What happens if in Exercise 2 you replace the tensor powers $V^{\otimes n}$ by the symmetric powers $Sym^n(V)$? What if you replace it by the exterior powers $\wedge^n(V)$?