Problem set 12 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Continuing the discussion from Problem set 11. Recall that we formulated some conditions on partitions λ of n. One of these was:

(S3') For every tableau t of type λ with increasing numbers down the columns, if $R \cap C_t = \{1\}$, then $t = r(t_0)$ with $r \in R$.

Exercise 6 from Problem set 11. Prove (S3').

Answer to Exercise 6. In class on Tuesday, November 21 we argued as follows. Let t be a tableau of type λ with increasing numbers down the columns and such that $R \cap C_t = \{1\}$. Suppose that an entry a from the last row of t_0 does not appear in the last row of t. Then let b be the last entry of the column of t in which a appears. Since a is not in the last row of t we see that b is not equal to a. Hence $b > a$ because in t the numbers are increasing down the columns. It follows that b must be in the last row of t_0 also. But then the transposition $\tau = (ab)$ is inh $R \cap C_t$, contradiction!

Thus the last rows of t and t_0 have the same entries. Then we can remove these last rows and reduce to a smaller n and we win by induction (details omitted).

The irreducible representations of S_n. By the solution of Exercise 6 above and the discussion in Problem set 11, for any partition λ of n we get a unique irreducible representation S^λ of S_n which occurs both in

$$M^\lambda = \mathrm{Ind}_{S_\lambda}^{S_n} 1 = \mathrm{Ind}_{R_{t_0}}^{S_n} 1 = \mathrm{Ind}_{R}^{S_n} 1$$

and in $\mathrm{Ind}_C^{S_n}(\epsilon)$ where $C = C_{t_0}$. (For notation please see Problem set 11.) Also, we know that S^λ occurs with multiplicity 1 in both of these.

Exercise 7. Let λ and μ be partitions of n. Prove the following statement: If $\mathrm{Hom}_{S_n}(S^\lambda, M^\mu) \neq 0$, then there exists a tableau t' of type μ such that every element of $C \cap R_{t'}$ is even.

Answer to Exercise 7. We will discuss this in the lecture on Tuesday Nov 28. Observe that

$$\mathrm{Hom}_{S_n}(S^\lambda, M^\mu) \neq 0 \Rightarrow \mathrm{Hom}_{S_n}(\mathrm{Ind}_C^{S_n}(\epsilon), M^\mu) \neq 0$$

By adjunction we have

$$\mathrm{Hom}_{S_n}(\mathrm{Ind}_C^{S_n}(\epsilon), M^\mu) = \mathrm{Hom}_C(\epsilon, \mathrm{Res}_C^{S_n} M^\mu)$$

Let R' be the row stabilizer of the basic tableau t'_0 of type μ. Then by Mackey's second theorem we have

$$\mathrm{Res}_C^{S_n} M^\mu) = \mathrm{Res}_C^{S_n} \mathrm{Ind}_{R'}^{S_n} 1 = \bigoplus \mathrm{Ind}_{C \cap R_{t'}}^{C} 1$$

where the direct sum is over a certain list of tableaux t' of type μ with row stabilizer $R_{t'}$. Namely, Mackey's theorem tells us to write $S_n = C\sigma_1 R' \amalg \ldots \amalg C\sigma_l R'$ and then have the direct sum of the inductions of 1 from $C \cap \sigma_i R' \sigma_i^{-1}$ to C. But then $\sigma_i R' \sigma_i^{-1} = R_{t'}$ where $t' = \sigma_i(t'_0)$ as desired. Thus $\mathrm{Hom}_{S_n}(S^\lambda, M^\mu) \neq 0$ implies that for some tableau t' of type μ we have

$$\mathrm{Hom}_C(\epsilon, \mathrm{Ind}_{C \cap R_{t'}}^{C} 1) \neq 0$$

By adjunction this says that

$$\mathrm{Hom}_{C\cap R_{t'}}(\epsilon, 1) \neq 0$$

which is clearly equivalent to the statement in the exercise.

Exercise 8. Prove the following statement: If $\mathrm{Hom}_{S_n}(S^\lambda, M^\mu) \neq 0$, then $\lambda \trianglerighteq \mu$.

Answer to Exercise 8. By Exercise 7 there exists a tableau t' of type μ such that every element of $C \cap R_{t'}$ is even. In particular there does not exist a transposition in $C \cap R_{t'}$. Let t_0 be the basic tableau of type λ as above. Now if $a \neq b$ are two entries in the same row of t', then they cannot be in the same column of t_0 (otherwise the transposition of a and b would be in the group $C \cap R_{t'}$). Hence by the Dominance lemma (see notes of Bob on reps of symmetric groups; we will discuss this in the lecture on Tuesday, Nov 28), we get $\lambda \trianglerighteq \mu$ as desired.

The irreducible representations of S_n. We conclude in particular that if $S^\lambda \cong S^\mu$ then both $\lambda \trianglerighteq \mu$ and $\mu \trianglerighteq \lambda$. Hence $\lambda = \mu$. Thus we get exactly the right number of irreducible representations of S_n.

Grothendieck group of an abelian category. Let $\mathcal{A}$ be an abelian category. This is basically any additive category (morphisms can be added and we have direct sums) and where we have a notion of short exact sequences (kernels and cokernels exist and $\mathrm{Im} \cong \mathrm{Coim}$). You won't need to know what this notion is in general because we only care about what happens in the examples. Then the Grothendieck group of $\mathcal{A}$ is the (unique up to unique isomorphism) abelian group $K(\mathcal{A})$ which comes with a map

$$\text{Objects of } \mathcal{A} \to K(\mathcal{A}), \quad M \longmapsto [M]$$

such that any relation among the images of this map is a consequence of the following elementary relations

(1) If M and M' are isomorphic objects of $\mathcal{A}$, then $[M] - [M']$ is zero in $K(\mathcal{A})$.
(2) If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence in $\mathcal{A}$, then $[M] - [M'] - [M'']$ is zero in $K(\mathcal{A})$.

In particular, if M is isomorphic to $M' \oplus M''$, then $[M] = [M'] + [M'']$ in $K(\mathcal{A})$.

Exercise 1. Show that every element of $K(\mathcal{A})$ can be represented as a difference $[M] - [M']$ for some objects M and M' of $\mathcal{A}$.

Exercise 2. Let G be a finite group. Let $\mathcal{A}$ be the category of representations of G (as in the course). Describe $K(\mathcal{A})$.

Exercise 3. Let $\mathcal{A}$ be the category of finite abelian groups. Describe $K(\mathcal{A})$.

Exercise 4. Let $\mathcal{A}$ be the category of finitely generated abelian groups. Describe $K(\mathcal{A})$.

Exercise 5. Let G be a finite group. Let $\mathcal{A}$ be the category of representations of G (as in the course). Show that tensor product on the category $\mathcal{A}$ defines a commutative ring structure on $K(\mathcal{A})$ (explain what this means and then prove it works).

Optional exercise A. Is there a "natural" way to define a ring structure on $K(\mathcal{A})$ in the examples of Exercise 3 or 4?

Optional exercise B. Let k be a field and let $\mathcal{A}$ be the category of all k-vector spaces. Show that $K(\mathcal{A})$ is zero.