Problem set 2 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Let k be a finite field with q elements. If you like, you may take $k = \mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ where p is a prime number and take $q = p$. Let V be a vector space of dimension $n \geq 1$ over k.

(1) Explain why $GL(V)$ is a finite group.
(2) Compute the order of $GL(V)$ in terms of n and q.

Denote $SL(V)$ the subgroup $GL(V)$ consisting of elements whose determinant is 1.

(3) Explain why $SL(V)$ is a normal subgroup of $GL(V)$.
(4) Describe the group $GL(V)/SL(V)$.
(5) How many elements does $SL(V)$ have?

Exercise 2. Let G be a finite group. In each of the following cases, explain briefly why there does not exist a finite dimensional representation π of G with character χ_π having the stated properties:

(1) $\chi_\pi(1) = -1$ where $1 \in G$ is the identity element.
(2) $\chi_\pi(1) = 1/2$,
(3) $\chi_\pi(1) = 5$ and $\chi_\pi(g) = 6$ for some $g \in G$,
(4) $\chi_\pi(1) = 2$ and $\chi_\pi(g) = 1/11$ for some $g \in G$,
(5) $\chi_\pi(g) = 4$ and $\chi_\pi(g^{-1}) = -4$ for some $g \in G$.

Exercise 3. Let G be a finite group. Let X be a finite set. Let $G \times X \to X$, $(g, x) \mapsto g \cdot x$ be an action of G on X. Let $\mathbf{C}[X]$ be the corresponding permutation representation of G. What this means is this:

(a) as a vector space $\mathbf{C}[X] = \{\text{maps } f : X \to \mathbf{C}\}$
(b) for $f \in \mathbf{C}[X]$ and $g \in G$ we define $g \cdot f$ by the rule

$$(g \cdot f)(x) = f(g^{-1} \cdot x)$$

for all $x \in X$.

Carefully explain why

(1) the inverse in the formula is necessary,
(2) the delta functions $\delta_x \in \mathbf{C}[X]$ where $x \in X$ form a basis for $\mathbf{C}[X]$, and
(3) $g \cdot \delta_x = \delta_{g \cdot x}$.

Remark. Often people think of elements of $\mathbf{C}[X]$ as formal linear sums $\xi = \sum t_x x$ with $t_x \in \mathbf{C}$. In other words, they think of $\mathbf{C}[X]$ as a $\mathbf{C}$-vector space with basis given by the elements of X. Then they define the G-action by the rule $g \cdot \xi = \sum t_x g \cdot x$ This version is isomorphic to ours in the exercise above, via the maps sending the element $\xi = \sum t_x x$ to the function $f = \sum t_x \delta_x$.

Exercise 4. Let us call a representation isomorphic to one of the representations of Exercise 3 a permutation representation.

(1) Give an example of a finite group G and a (finite dimensional as always) representation V which is not a permutation representation.
(2) Show that if V_1 and V_2 are permutation representations of the same finite group G, then so is $V_1 \oplus V_2$.
(3) Show that if V_1 and V_2 are permutation representations of the same finite group G, then so is $V_1 \otimes V_2$.

(4) Give an example of a group G and a permutation representation V such that $\wedge^2(V)$ is not a permutation representation. (If you solve this, then you've solved part (1) as well.)

(5) Show that a permutation representation is isomorphic to its dual. Hint: you may use that a representation V is isomorphic to its dual if and only if there exists a G-invariant nondegenerate bilinear pairing $\langle,\rangle : V \times V \to \mathbf{C}$.

Exercise 5. Let $n \geq 1$. Let $\zeta = \exp(2i\pi/n)$ be the usual primitive nth root of 1. Consider the $n \times n$ matrix

$$A = \mathrm{diag}(1, \zeta, \zeta^2, \dots) = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & \zeta & 0 & \dots \\ 0 & 0 & \zeta^2 & \dots \\ \dots & & & \end{pmatrix}$$

and the permutation matrix corresponding to the n-cycle $(12\dots n)$, namely

$$B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & & \\ 0 & 1 & 0 & \dots & & \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

Prove that

(1) A and B generate a finite subgroup G of $GL_n(\mathbf{C})$
(2) the representation of G on $\mathbf{C}^n$ you get in this way is irreducible.