Problem set 3 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Let G be a finite group and let $H \subset G$ be a subgroup. Let (V, π) be a representation of G. Denote (V, π') be the representation of H where $\pi' : H \to GL(V)$ is the restriction of π to H.

(1) Show that if (V, π') is irreducible, then (V, π) is irreducible.
(2) Give an example where (V, π) is irreducible, but (V, π') is not irreducible.

Exercise 2. Let G and H be finite groups. Let $G \times H$ be the product group.

(1) Express the number of conjugacy classes of $G \times H$ in terms of the number of conjugacy classes of G and H.
(2) Let (V, π) be a representation of G and let (W, θ) be a representation of H. Then $V \otimes W$ is a representation of $G \times H$ by letting (g, h) act by $\pi(g) \otimes \theta(h)$. Show that if V and W are irreducible, then $V \otimes W$ is irreducible. Hint: compute $(\chi_{V\otimes W}, \chi_{V\otimes W})$.
(3) Prove every irreducible representation of $G \times H$ is of this form. Hint: count!

Remark. The results of Exercise 2 hold for arbitrary groups and finite dimensional representations; there is a way to do the exercise without using the hints.

Exercise 3. Let p be a prime number. Let $1 < i < p$ be a generator of the multiplicative group $\mathbf{F}_p^*$ of the field $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ (the group $\mathbf{F}_p^*$ is always cyclic). Let

$$G = \langle a, b \text{ with relations } b^p = 1,\ aba^{-1} = b^i,\ a^{p-1} = 1\rangle$$

Then G is isomorphic to the semi-direct product $G = \mathbf{F}_p \rtimes \mathbf{F}_p^*$ where $\mathbf{F}_p^*$ acts on $\mathbf{F}_p$ by multiplication.

(1) Show that if (V, π) is a nonzero irreducible representation, then either $\dim(V) = 1$ or $\dim(V) \geq p - 1$. Hint: look at the eigenvalues of b.
(2) How many 1-dimensional irreducible characters does G have?
(3) Prove that G has a unique irreducible representation of dimension $p - 1$ besides the 1-dimensional ones found above. Hint: You can count conjugacy classes or (easier) you can use a formula we discussed in the lectures.

Exercise 4. Let G be the group of order 12 generated by elements a and b subject to the relations

$$a^6 = 1,\ a^3 = b^2,\ b^{-1}ab = a^{-1}$$

It follows from these relations that every element of G can be uniquely written as $a^r b^s$ with $0 \leq r \leq 5$ and $0 \leq s \leq 1$. Give the character table of G.

Exercise 5. A representation (V, π) of a group G is said to be *faithful*, if the homomorphism $\pi : G \to GL(V)$ is injective. Give an example of a finite group G which does not have a faithful representation of dimension ≤ 2023. Hint: look at what happened in Exercise 3.