Problem set 4 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. The point of this exercise is for you to read this material and think about it; please do not worry about providing a lot of details in your answers. Let $\phi : R \to S$ be a homomorphism of possibly noncommutative rings (associative and unital). Let Mod_R and Mod_S be the category of left R-modules and left S-modules. Consider the functors

$$\begin{aligned} res &: \mathrm{Mod}_S \to \mathrm{Mod}_R, \quad N \mapsto N_R \\ ind &: \mathrm{Mod}_R \to \mathrm{Mod}_S, \quad M \mapsto S \otimes_R M \\ coind &: \mathrm{Mod}_R \to \mathrm{Mod}_S, \quad M \mapsto \mathrm{Hom}_R(S, M) \end{aligned}$$

The first functor, called *restriction*, assigns to an S-module N the R-module N_R which has the same underlying abelian group and multiplication given by $r \cdot n = \phi(r)n$. The second functor, called *induction*, assigns to an R-module M the S-module $S \otimes_R M$. The third functor, called *co-induction*, assigns to an R-module M the group of left R-module maps $\mathrm{Hom}_R(S, M)$ with left S-module structure given by $(s \cdot \lambda)(s') = \lambda(s's)$.

(1) Prove that *ind* is a left adjoint to restriction, i.e., prove that for an R-module M and an S-module N we have a canonical isomorphism

$$\mathrm{Hom}_R(M, res(N)) = \mathrm{Hom}_S(ind(M), N)$$

by indicating how an element of the left side produces an element of the right side and vice versa; don't check that these assignments produce inverse maps.

(2) Prove that *coind* is a right adjoint to restriction, i.e., prove that for an R-module M and an S-module N we have a canonical isomorphism

$$\mathrm{Hom}_R(res(N), M) = \mathrm{Hom}_S(N, coind(M))$$

by indicating how an element of the left side produces an element of the right side and vice versa; don't check that these assignments produce inverse maps.

(3) Prove that if S has a right R-basis $\{s_i\}_{i\in I}$, so $S = \bigoplus s_i R$, then

$$ind(M) = S \otimes_R M = \bigoplus\nolimits_{i\in I} s_i M \quad \text{(formal sum)}$$

with left S-module structure given by $ss_i m = \sum_j s_j a_{ji} m$ if $ss_i = \sum_j s_j a_{ji}$ with $a_{ji} \in R$.

(4) Prove that if S has a **finite** left R-basis $\{t_i\}_{i\in I}$, so $S = \bigoplus Rt_i$, then

$$coind(M) = \mathrm{Hom}_R(S, M) = \bigoplus\nolimits_{i\in I} M t_i^\vee \quad \text{(formal sum)}$$

with left S-module structure given by $smt_i^\vee = \sum_j b_{ij} m t_j^\vee$ if $t_i s = \sum_j b_{ij} t_j$ with $b_{ij} \in R$.

(5) Show that under the assumptions of (4) we have a functorial isomorphism

$$coind(M) = \mathrm{Hom}_R(S, R) \otimes_R M$$

for M in Mod_R.

(6) Under the assumptions of (4) conclude that the functors *ind* and *coind* are isomorphic if and only if S and $\mathrm{Hom}_R(S, R)$ are isomorphic as (S, R)-bimodules, i.e., there is a group isomorphism $\alpha : S \to \mathrm{Hom}_R(S, R)$ such that $s\alpha(s')r = \alpha(ss'r)$ for all $s, s' \in S$ and $r \in R$.

(7) Give an example of $R \to S$ as in (4) such that *ind* and *coind* are not isomorphic.

Exercise 2. Let p be an odd prime number. Let $k = \mathbf{F}_p$ be the field with p elements. Let $G = PGL_2(k)$; this group is the quotient of $GL_2(k)$ by the subgroup of invertible scalar matrices, i.e., we have a short exact sequence

$$1 \to k^* \to GL_2(k) \to G \to 1$$

of finite groups.

(1) Let b and a be the elements of G which are the image of

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$$

for $i \in k^*$. Express aba^{-1} in terms of b and i.

(2) Conclude that G contains a subgroup isomorphic to the semidirect product $k \rtimes k^*$ discussed in problem set 3.

(3) Show that the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is a commutator in $GL_2(k)$ (this uses that p is odd). Hint: use calculations from (1).

(4) Show that the elements

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

in $GL_2(k)$ are conjugate.

(5) For any $\mu \in k \setminus \{0, 1\}$ show that there exist nonzero $x, y \in k$ such that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \quad \text{is conjugate to} \quad \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$$

in $GL_2(k)$. Hint: use linear algebra to express this in terms of traces.

(6) Recall that $k^*/(k^*)^2$ is a group of order 2 (this uses that p is odd). Show that there is a canonical surjective map $c : G \to k^*/(k^*)^2$ by sending the class of a matrix to the determinant modulo squares.

(7) Show that the commutator subgroup of $GL_2(k)$ contains all elements of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

for $x, y \in k$, $\mu \in k^*$.

(8) How many elements do you get this way?

(9) Show that the commutator subgroup G' of G is the kernel of c. Hint: it suffices to show that G' contains more than $|G|/3$ elements. Count using the above.

(10) Show that G' is generated by the conjugacy class of b. (This should be clear from your answer above.)

(11) Besides the two 1-dimensional representations coming from homomorphisms $G/G' \to \mathbf{C}^*$, show that every other nonzero irreducible representation V of G has dimension $\dim(V) \geq p-1$. Hint: argue that b cannot be mapped to the identity of V and compare with the exercise from last homework.
(12) Denote $\mathbf{P}^1(k)$ the set of nonzero vectors in $k^2 = k \oplus k$ up to scaling. So $\mathbf{P}^1(k) = (k^2 \setminus \{(0,0)\})/k^*$. Show that $\mathbf{P}^1(k)$ has $p+1$ elements.
(13) Show that there is a natural action of G on $\mathbf{P}^1(k)$.
(14) Show this action is doubly transitive.
(15) Conclude that G has at least one irreducible representation of dimension p.