Problem set 6 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Let G be a finite group. Prove the following are equivalent

(1) G is a simple group, and
(2) for every non-trivial irreducible character χ of G and all $g \in G$, $g \neq 1$ we have $\chi(g) \neq \chi(1)$.

Exercise 2. Let H and G be finite groups. Let $f_1, f_2 : H \to G$ be two group homomorphisms. If (V, π) is a representation of G, then restricting along f_1 we get a representation $(V, \pi \circ f_1)$ of H and restricting along f_2 we get a representation $(V, \pi \circ f_2)$ of H.

(1) Give an example to show that it may happen that $(V, \pi \circ f_1)$ of H and $(V, \pi \circ f_2)$ are not isomorphic as H-representations.
(2) Suppose that there exists a $g \in G$ such that $f_1(h) = gf_2(h)g^{-1}$ for all $h \in H$. Show that $(V, \pi \circ f_1)$ of H and $(V, \pi \circ f_2)$ are isomorphic as H-representations.

Exercise 3. Let $H \subset G$ be a subgroup of a finite group G. Let (W, ρ) be a permutation representation of H. Show that $\mathrm{Ind}_H^G(W)$ is a permutation representation of G.

Exercise 4. Let G be a finite group. Let (V, π) be a representation of G. We say (V, π) is a *monomial representation* if there exists a basis $e_1, \ldots, e_n$ of V such that for each $g \in G$ the map $\pi(g)$ permutes the e_i up to scalars. In other words, for each $g \in G$ and $1 \leq i \leq n$ there exists a $j \in \{1, \ldots, n\}$ and a $\lambda \in \mathbf{C}^*$ such that $\pi(g)(e_i) = \lambda e_j$. Show that a monomial representation is a direct sum of representations of the form $\mathrm{Ind}_H^G(\chi)$ where $H \subset G$ is a subgroup and $\chi : H \to \mathbf{C}^*$ is a character.