Problem set 7 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Let G be a finite group. Let V be a finite dimensional $\mathbf{C}$-vector space. Let

$$\pi_t : G \longrightarrow GL(V), \quad t \in [0,1]$$

be a continuous family of representations of G. This means that for each $t \in [0,1]$ the map π_t is a homomorphism and that for each fixed $g \in G$ the function $[0,1] \to GL(V)$, $t \mapsto \pi_t(g)$ is continuous[1]. Prove that π_0 and π_1 are isomorphic representations of G.

Exercise 2. Let p be a prime number. Let G be the group

$$G = \langle a_1, a_2, b_1, b_2, c\rangle / relations$$

where the relations are

(1) all of the elements a_1, a_2, b_1, b_2, c have order p,
(2) c is central,
(3) the pairs a_1, a_2 and a_1, b_2 and a_2, b_1 and b_1, b_2 commute,
(4) $a_1 b_1 a_1^{-1} b_1^{-1} = c$ and $a_2 b_2 a_2^{-1} b_2^{-1} = c$.

You may use that in this group every element has a unique expression of the form $a_1^{i_1} a_2^{i_2} b_1^{j_1} b_2^{j_2} c^k$ with $0 \leq i_1, i_2, j_1, j_2, k < p$. Construct all irreducible representations of G using the method discussed in the lectures.

Exercise 3. Let p be an odd prime number. Let $k = \mathbf{F}_p$ be the field with p elements. Let $G = GL_2(k)$. Let $B \subset G$ be the Borel subgroup, i.e.,

$$B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

Choose a character $\chi_0 : k^* \to \mathbf{C}^*$ and define a character $\chi : B \longrightarrow \mathbf{C}^*$ by the rule

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \longmapsto \chi_0(a)$$

Compute the character of $\mathrm{Ind}_B^G(\chi)$.

[1] Here $[0,1] \subset \mathbf{R}$ is the closed interval. Choosing a basis of V we may think of π_t as a homomorphism into $GL_n(\mathbf{C})$ where $n = \dim(V)$, so the condition is that the map $[0,1] \to GL_n(\mathbf{C})$, $t \mapsto \pi_t(g)$ is continuous, which in turn means that the matrix coefficients $\pi_t(g)_{ij}$ are continuous maps $[0,1] \to \mathbf{C}$.