Problem set 8 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Let G be a finite group. Recall that given an class function f on G we let $\psi_2(f)$ be the class function sending g to $f(g^2)$. Let (V, π) be a representation of G. Recall that In the proof of Theorem 3.2 in the notes on real representations we saw that

$$\psi_2(\chi_V) = \chi_{Sym^2(V)} - \chi_{\wedge^2(V)}$$

(1) What does it mean in terms of representations if $\psi_2(\chi_V) = \chi_W$ for some representation (W, ρ) of G?
(2) Give an example of a finite group G and an irreducible (V, π) of dimension > 1 such that there exists a (W, ρ) as in (1).
(3) Give an example of a finite group G and an irreducible (V, π) of dimension > 1 where there does not exist a (W, ρ) as in (1).
(4) Show that if for every irreducible (V, π) there exists a (W, ρ) as in (1), then the same is true for every representation (V, π) of G.
(5) For a class function f on G define $\psi_3(f)$ to be the class function sending g to $f(g^3)$. Can you find an expression for $\psi_3(\chi_V)$ in terms of characters of representations of G?

Exercise 2. Let $a_0, a_1, \ldots$ be a series of integers. Suppose that there exist integers $c, n > 0$ such that for all $i = 0, 1, \ldots, n-1$ there exists a polynomial $P_i \in \mathbf{Q}[x]$ with $a_{i+nk} = P_i(k)$ for $k \geq c$. Explain why $\sum_k a_k t^k$ is a rational function of t.

Remark. Let A be a commutative, graded $\mathbf{C}$-algebra which is finitely generated as a $\mathbf{C}$-algebra and such that $\dim_{\mathbf{C}} A_0 < \infty$. Then the sequence of numbers $a_k = \dim_{\mathbf{C}} A_k$ satisfies the assumptions and hence the conclusions of Exercise 2 above.

Exercise 3. Consider the commutative, graded $\mathbf{C}$-algebra

$$A = \mathbf{C}[x_1, \ldots, x_r]/(f)$$

where $x_1, \ldots, x_r$ are homogenous of degrees $d_1, \ldots, d_r$ and f is nonzero and homogenous of degree e. Compute the Poincaré series $P(A, t) = \sum_k \dim_{\mathbf{C}} A_k$ as a rational function of t. An example is $\mathbf{C}[x, y, z]/(x^{15} + y^{10} + z^6)$ where $\deg(x) = 2$, $\deg(y) = 3$, and $\deg(z) = 5$. Feel free to only work out the exercise in the example case.

Exercise 4. Let G be a finite group. Let us consider the condition on the group

(*) for every representation (V, π) of G there exists a representation (W, ρ) of G such that $\psi_2(\chi_V) = \chi_W$.

We studied this condition in Exercise 1. My guess while writing these exercises is that (*) holds if $|G|$ is odd. Can you say anything about this? For example, take the group G of order p^5 we considered in the previous exercise set (or you can take the similar group of order p^3 we discussed in the lecture that I called a "Heisenberg" group). Does (*) hold for G?