Real representations

1 Definition of a real representation

Definition 1.1. Let $V_{\mathbb{R}}$ be a finite dimensional real vector space. A *real representation* of a group G is a homomorphism $\rho_{V_{\mathbb{R}}}\colon G \to \operatorname{Aut} V_{\mathbb{R}}$, where $\operatorname{Aut} V_{\mathbb{R}}$ denotes the $\mathbb{R}$-linear isomorphisms from $V_{\mathbb{R}}$ to itself. Homomorphisms and isomorphisms of real representations are defined in the obvious way. After a choice of basis, a real representation is equivalent to a homomorphism $\rho\colon G \to GL(n, \mathbb{R})$, and two such homomorphisms ρ_1 and ρ_2 are isomorphic real representations $\iff$ they are conjugate in $GL(n, \mathbb{R})$, i.e. $\iff$ there exists an $A \in GL(n, \mathbb{R})$ such that $\rho_2(g) = A\rho_1(g)A^{-1}$ for all $g \in G$.

Because $GL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{C})$, every real representation $V_{\mathbb{R}}$ defines a (complex) representation V. More abstractly, given a real vector space $V_{\mathbb{R}}$, we define its *complexification* to be the tensor product $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Concretely, think of $\mathbb{R}^n$ being enlarged to $\mathbb{C}^n$. For any real vector space $V_{\mathbb{R}}$, if $v_1, \dots, v_n$ is a basis for $V_{\mathbb{R}}$, then by definition $V_{\mathbb{R}}$ is the set of all linear combinations of the v_i with real coefficients, and V is the set of all linear combinations of the v_i with complex coefficients. In particular, $v_1, \dots, v_n$ is a basis for the complex vector space V. Conversely, given a (complex) vector space V and a basis $v_1, \dots, v_n$ of V, we can define a vector subspace $V_{\mathbb{R}}$ of V by taking the set of all linear combinations of the v_i with real coefficients, and V is then the complexification of $V_{\mathbb{R}}$. We have an inclusion $\operatorname{Aut} V_{\mathbb{R}} \to \operatorname{Aut} V$, which can be summarized by the commutative diagram

$$\begin{tikzcd}
\operatorname{Aut} V_{\mathbb{R}} \arrow[r] \arrow[d, "\cong"'] & \operatorname{Aut} V \arrow[d, "\cong"] \\
GL(n, \mathbb{R}) \arrow[r] & GL(n, \mathbb{C}),
\end{tikzcd}$$

where the vertical isomorphisms correspond to the choice of basis $v_1, \dots, v_n$. However, the top horizontal inclusion is canonical, i.e. does not depend on the choice of basis.

Definition 1.2. A representation $\rho_V \colon G \to \operatorname{Aut} V$ can be *defined over* $\mathbb{R}$ if there exists a real vector space $V_{\mathbb{R}}$ and a real representation $\rho_{V_{\mathbb{R}}} \colon G \to \operatorname{Aut} V_{\mathbb{R}}$ such that V is the complexification of $V_{\mathbb{R}}$ and ρ_V is the image of $\rho_{V_{\mathbb{R}}}$ via the inclusion $\operatorname{Aut} V_{\mathbb{R}} \to \operatorname{Aut} V$. Equivalently, there exists a basis of V such that, for every $g \in G$, the matrices $\rho_V(g)$ have real entries.

Remark 1.3. (1) We can make the same definition for any subfield K of $\mathbb{C}$, for example for $K = \mathbb{Q}$.

(2) Every complex vector space V of dimension n is also a real vector space of dimension $2n$, by only allowing scalar multiplication by real numbers. To see the statement about the dimensions, if $v_1, \dots, v_n$ is a basis of V as a complex vector space, then it is easy to check that $v_1, iv_1, v_2, iv_2, \dots, v_n, iv_n$ is a basis of V viewed as a real vector space. Since every complex linear isomorphism is automatically real liner, there is a homomorphism $GL(n, \mathbb{C}) \to GL(2n, \mathbb{R})$ which is a little messy to write down in general. For $n = 1$, it corresponds to the homomorphism $\varphi \colon \mathbb{C}^* \to GL(2, \mathbb{R})$ defined by $\varphi(a + bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

The following gives a *necessary* condition for a representation of a finite group to be defined over $\mathbb{R}$:

Lemma 1.4. *If ρ_V can be defined over $\mathbb{R}$, then, for all $g \in G$, $\chi_V(g) \in \mathbb{R}$. More generally, if K is a subfield of $\mathbb{C}$ and ρ_V can be defined over K, then, for all $g \in G$, $\chi_V(g) \in K$.*

Proof. This is clear since the trace of an $n \times n$ matrix with entries in K is an element of K. $\square$

As we shall see, the necessary condition above is not in general sufficient.

2 When is an irreducible representation defined over $\mathbb{R}$

We begin by analyzing the condition that $\chi_V(g) \in \mathbb{R}$ for all $g \in G$, and shall only consider the case of an irreducible representation in what follows.

Lemma 2.1. *If V is a G-representation of the finite group G, then $\chi_V(g) \in \mathbb{R}$ for all $g \in G \iff V \cong V^*$. If moreover V is irreducible and $V \cong V^*$, then there exists a nonzero $\varphi \in \operatorname{Hom}^G(V, V^*)$ and it is unique up to a nonzero scalar, i.e.* $\dim \operatorname{Hom}^G(V, V^*) = 1$.

Proof. Since $\chi_{V^*} = \overline{\chi_V}$, we see that $\chi_V(g) \in \mathbb{R}$ for all $g \in G \iff \chi_V = \overline{\chi_V} \iff \chi_V = \chi_{V^*} \iff V \cong V^*$. The remaining statement then follows from Schur's lemma. $\square$

We define $\mathrm{Bil}(V)$ to be the set of bilinear functions $F\colon V \times V \to \mathbb{C}$. General results about tensor products tell us that

$$\mathrm{Bil}(V) \cong (V \otimes V)^* \cong V^* \otimes V^* \cong \mathrm{Hom}(V, V^*).$$

However, we will explicitly construct the isomorphism $\mathrm{Bil}(V) \cong \mathrm{Hom}(V, V^*)$:

Lemma 2.2. *The map $A\colon \mathrm{Bil}(V) \to \mathrm{Hom}(V, V^*)$ defined by*

$$A(F)(v)(w) = F(v, w)$$

is an isomorphism of vector spaces. If V is a G-representation and we define

$$\rho_{\mathrm{Bil}(V)}(g)(F)(v, w) = F(\rho_V(g)^{-1}(v), \rho_V(g)^{-1}(w)),$$

then A is a G-isomorphism, where as usual, given $\varphi \in \mathrm{Hom}(V, V^*)$,

$$\rho_{\mathrm{Hom}(V,V^*)}(g)(\varphi)(v)(w) = \varphi(\rho_V(g)^{-1}v)(\rho_V(g)^{-1}(w).$$

Proof. If we define $A(F)$ as in the statement, then it is easy to see that $A(F)(v)$ is linear in w and that $v \mapsto A(F)(v)$ is linear in v, so that $A(F) \in \mathrm{Hom}(V, V^*)$. Also, a short computation shows that $A(F_1 + F_2) = A(F_1) + A(F_2)$ and that $A(tF) = tA(F)$, so A is a linear map of vector spaces. To show that A is an isomorphism, we define an inverse function: let $B\colon \mathrm{Hom}(V, V^*) \to \mathrm{Bil}(V)$ be defined by

$$B(\varphi)(v, w) = \varphi(v)(w).$$

Again, an easy calculation shows that $B \circ A = \mathrm{Id}$, $A \circ B = \mathrm{Id}$. Finally, if V is a G-representation, then

$$\begin{aligned} A(\rho_{\mathrm{Bil}(V)}(g)(F))(v)(w) = \rho_{\mathrm{Bil}(V)}(g)(F)(v, w) &= F(\rho_V(g)^{-1}(v), \rho_V(g)^{-1}(w)) \\ &= \rho_{\mathrm{Hom}(V,V^*)}(g)(A(F))(v)(w), \end{aligned}$$

so that A is a G-morphism and hence a G-isomorphism. $\square$

Corollary 2.3. *If V is an irreducible G-representation, then $\dim(\mathrm{Bil}(V)^G)$ is 0 if V is not isomorphic to V^* and 1 if $V \cong V^*$.* $\square$

To analyze $\mathrm{Bil}(V)$ further, we make the following definition:

Definition 2.4. Let $F \in \mathrm{Bil}(V)$. Then F is *symmetric* if $F(v,w) = F(w,v)$ for all $v, w \in V$, and F is *antisymmetric* if $F(v,w) = -F(w,v)$ for all $v, w \in V$. Let $\mathrm{Sym}^2 V^*$ be the set of all symmetric $F \in \mathrm{Bil}(V)$ and let $\bigwedge^2 V^*$ denote the set of all antisymmetric $F \in \mathrm{Bil}(V)$. Clearly both $\mathrm{Sym}^2 V^*$ and $\bigwedge^2 V^*$ are vector subspaces of $\mathrm{Bil}(V)$. If V is a G-representation, so that $\mathrm{Bil}(V)$ is also a G-representation, then from the definition of $\rho_{\mathrm{Bil}(V)}$ it is easy to see that $\mathrm{Sym}^2 V^*$ and $\bigwedge^2 V^*$ are G-invariant subspaces of $\mathrm{Bil}(V)$.

Lemma 2.5. $\mathrm{Bil}(V) = \mathrm{Sym}^2 V^* \oplus \bigwedge^2 V^*$. *If V is a G-representation, then the above is a direct sum of G-invariant subspaces.*

Proof. Define $\pi_1 \colon \mathrm{Bil}(V) \to \mathrm{Sym}^2 V^*$ and $\pi_2 \colon \mathrm{Bil}(V) \to \bigwedge^2 V^*$ by:

$$\pi_1(F)(v,w) = \frac{1}{2}(F(v,w) + F(w,v));$$
$$\pi_2(F)(v,w) = \frac{1}{2}(F(v,w) - F(w,v)).$$

Then clearly $\pi_1(F) = F \iff F \in \mathrm{Sym}^2 V^*$, $\pi_1(F) = 0 \iff F \in \bigwedge^2 V^*$, and similarly $\pi_2(F) = 0 \iff F \in \mathrm{Sym}^2 V^*$, $\pi_1(F) = F \iff F \in \bigwedge^2 V^*$. Also $\pi_1 + \pi_2 = \mathrm{Id}$. It then follows that $\mathrm{Bil}(V) = \mathrm{Sym}^2 V^* \oplus \bigwedge^2 V^*$. The last statement is then a general fact. $\square$

Corollary 2.6. *Let V be an irreducible representation. If V and V^* are not isomorphic, then* $\mathrm{Bil}(V)^G = 0$. *If V and V^* are isomorphic, then either* $\dim(\mathrm{Sym}^2 V^*)^G = 1$ *and* $\bigwedge^2 V^* = 0$ *or* $\dim(\bigwedge^2 V^*)^G = 1$ *and* $\mathrm{Sym}^2 V^* = 0$. $\square$

We can now state the main result concerning real representations:

Theorem 2.7. *Let V be an irreducible G-representation.*

(i) *V and V^* are not isomorphic* $\iff \mathrm{Bil}(V)^G = 0$.

(ii) *$V \cong V^*$ and V is defined over $\mathbb{R}$* $\iff \dim(\mathrm{Sym}^2 V^*)^G = 1$ *and* $\bigwedge^2 V^* = 0$.

(iii) *$V \cong V^*$ and V is not defined over $\mathbb{R}$* $\iff \dim(\bigwedge^2 V^*)^G = 1$ *and* $\mathrm{Sym}^2 V^* = 0$. *Moreover, in this case* $\dim V$ *is even.*

Proof. We have already seen (i). Also (ii) $\implies$ (iii), except for the last statement about the dimension, since (iii) is just the equivalence of the negations of the two statements of (ii). So we must prove (ii).

$\implies$: Suppose that V is defined over $\mathbb{R}$. In other words, there exists a basis $v_1, \dots, v_n$ of V such that the matrix of ρ_V with respect to this basis has real entries. Let $V_{\mathbb{R}}$ be the real span of the v_i:

$$V_{\mathbb{R}} = \left\{ \sum_{i=1}^{n} t_i v_i : t_i \in \mathbb{R} \right\}.$$

Thus $V_{\mathbb{R}}$ is a real vector subspace of V and ρ_V comes from a real representation $\rho_{V_{\mathbb{R}}}$ of G on $V_{\mathbb{R}}$, i.e. ρ_V is defined over $\mathbb{R}$. There exists a positive definite inner product (i.e. a symmetric $\mathbb{R}$-bilinear function) on $V_{\mathbb{R}}$, for example we could define

$$\left\langle \sum_{i=1}^{n} s_i v_i, \sum_{i=1}^{n} t_i v_i \right\rangle = \sum_{i=1}^{n} s_i t_i.$$

This inner product is not G-invariant, but we can make it G-invariant by averaging over G: define

$$F_{\mathbb{R}}(v, w) = \frac{1}{\#(G)} \sum_{g \in G} \langle \rho_{V_{\mathbb{R}}}(v), \rho_{V_{\mathbb{R}}}(w) \rangle.$$

Then $F_{\mathbb{R}}$ is symmetric and it is positive definite, because it is a sum of positive definite inner products. In particular $F_{\mathbb{R}} \neq 0$. Note that $F_{\mathbb{R}}$ is specified by its values $F_{\mathbb{R}}(v_i, v_j)$ and the G-invariance of $F_{\mathbb{R}}$ is equivalent to the statement that, for all i, j and all $g \in G$,

$$F_{\mathbb{R}}(\rho_{V_{\mathbb{R}}}(v_i), \rho_{V_{\mathbb{R}}}(v_j)) = F_{\mathbb{R}}(v_i, v_j).$$

Now we can extend $F_{\mathbb{R}}$ to a $\mathbb{C}$-bilinear function F on V, by defining

$$F(v, w) = \sum_{i,j} s_i t_j F_{\mathbb{R}}(v_i, v_j),$$

where $v = \sum_i s_i v_i$ and $w = \sum_i t_i w_i$. In particular, if $v, w \in V_{\mathbb{R}}$, then $F(v, w) = F_{\mathbb{R}}(v, w)$ so that $F \neq 0$. Moreover F is symmetric because $F_{\mathbb{R}}$ is symmetric, and hence $F_{\mathbb{R}}(v_i, v_j) = F_{\mathbb{R}}(v_j, v_i)$. Finally, one checks that F is G-invariant, which is equivalent to the statement that $F(\rho_{V_{\mathbb{R}}}(v_i), \rho_{V_{\mathbb{R}}}(v_j)) = F(v_i, v_j)$ and thus follows from the corresponding statement for $F_{\mathbb{R}}$. Thus F is a nonzero element of $\mathrm{Sym}^2 V^*)^G$. It follows that $\dim(\mathrm{Sym}^2 V^*)^G = 1$ and $\bigwedge^2 V^* = 0$.

$\impliedby$: We must use the existence of a nonzero $F \in \mathrm{Sym}^2 V^*)^G$ to show that V is defined over $\mathbb{R}$. We begin with a digression on complex structures. Let $V_{\mathbb{R}}$ be a real vector space with complexification V. We can think of this as follows: there exists a basis $v_1, \dots, v_n$ of V such that $V_{\mathbb{R}}$ is the real span of $v_1, \dots, v_n$. Thus $v_1, iv_1, v_2, iv_2, \dots, v_n, iv_n$ is a real basis of V and every $v \in V$ can be uniquely written as $w + iu$, where $w, u \in V_{\mathbb{R}}$. In other words, as real vector spaces,

$$V \cong V_{\mathbb{R}} \oplus iV_{\mathbb{R}}.$$

Now we can define conjugation γ on V:

$$\gamma(w + iu) = w - iu.$$

Thus γ is $\mathbb{R}$-linear with $\gamma^2 = \mathrm{Id}$ and $+1$-eigenspace $V_{\mathbb{R}}$ and -1-eigenspace $iV_{\mathbb{R}}$. . In terms of the basis $v_1, \dots, v_n$, if $v = \sum_i t_i v_i$ is a vector in V, then

$$\gamma(\sum_i t_i v_i) = \sum_i \bar{t}_i v_i.$$

Then γ is conjugate linear: $\gamma(tv) = \bar{t}\gamma(v)$. Finally, if $A \in \operatorname{End} V$ satisfies $A(V_{\mathbb{R}}) \subseteq V_{\mathbb{R}}$, i.e. the matrix of A with respect to the basis $v_1, \dots, v_n$ has real entries, then A commutes with γ since it preserves two eigenspaces, and conversely, if A commutes with γ, then $A(V_{\mathbb{R}}) \subseteq V_{\mathbb{R}}$ and hence the matrix of A with respect to the basis $v_1, \dots, v_n$ has real entries.

Conversely, suppose that V is a complex vector space and that $\gamma\colon V \to V$ is conjugate linear and hence $\mathbb{R}$-linear, and that $\gamma^2 = \mathrm{Id}$. Then γ is diagonalizable over $\mathbb{R}$, i.e. $V \cong V_+ \oplus V_-$, where $V_\pm$ are real vector subspaces of V, $\gamma|V_+ = \mathrm{Id}$ and $\gamma|V_- = -\mathrm{Id}$. In fact, we can define V_+ to be the $+1$-eigenspace of γ and V_- to be the -1-eigenspace. Then setting

$$\pi_+(v) = \frac{1}{2}(v + \gamma(v))$$
$$\pi_-(v) = \frac{1}{2}(v - \gamma(v)),$$

it is easy to check that $\operatorname{Im} \pi_\pm = V_\pm$, $\operatorname{Ker} \pi_\pm = V_\mp$, and $\pi_+ + \pi_- = \mathrm{Id}$, giving the direct sum decomposition. Moreover

$$v \in V_+ \iff \gamma(v) = v \iff \gamma(iv) = -iv \iff iv \in V_-.$$

Thus multiplication by i defines an isomorphism from V_+ to V_-. It follows that an $\mathbb{R}$-basis for V_+ is a $\mathbb{C}$-basis for V, and that V is the complexification of V_+. Finally, if $A \in \operatorname{End} V$ is complex linear and A commutes with γ, then $A(V_+) \subseteq V_+$. Hence A has real coefficients with respect to any basis of V which is a real basis of V_+. In particular, if $\rho_V\colon G \to \operatorname{Aut} V$ is a homomorphism and γ commutes with $\rho_V(g)$ for every $g \in G$, then ρ_V defines a real representation on V_+ and ρ_V is the complexification of this representation.

Returning to our situation, we have a nonzero symmetric G-invariant $F \in \operatorname{Sym}^2 V^*$, corresponding to a G-invariant homomorphism $\varphi\colon V \to V^*$, necessarily an isomorphism by Schur's lemma. Here $F(v,w) = \varphi(v)(w)$, so the symmetry condition is the statement that, for all $v, w \in V$,

$$\varphi(v)(w) = \varphi(w)(v).$$

There exists a positive definite Hermitian inner product on V, so after averaging there exists a G-invariant positive definite Hermitian inner product $H(v,w)$ on V. Such an H defines an $\mathbb{R}$-linear function $\psi\colon V \to V^*$ by the rule

$$\psi(v)(w) = H(w,v).$$

Note that we have to switch the order to make $\psi(v)$ is linear in w. However, ψ is conjugate linear in v. It is easy to see that ψ is an isomorphism: since V and V^* have the same dimension as real vector spaces, it suffices to show that ψ is injective, i.e. that $\psi(v)$ is not the zero element in V^* for $v \neq 0$. This follows since $\psi(v)(v) = H(v,v) > 0$.

Define $\alpha\colon V \to V$ by $\alpha = \psi^{-1} \circ \varphi$. Then α is conjugate linear since it is a composition of a complex linear and a conjugate linear map, and α is an isomorphism of real vector spaces. Finally, α is G-invariant since ψ and φ are G-invariant.

Consider $\alpha^2\colon V \to V$, which is complex linear as it is the composition of two conjugate linear maps. It is also a G-invariant isomorphism since it is the composition of two such. Thus, by Schur's lemma, $\alpha^2 = \lambda \operatorname{Id}$ for some nonzero complex number λ.

Claim 2.8. *λ is a positive real number.*

Proof. By the definition of ψ, $\psi(v)(w) = H(w,v)$ for all $v, w \in V$. Thus, for all $f \in V^*$,

$$f(w) = H(w, \psi^{-1}(f)).$$

If in addition $f = \varphi(v)$, this says that

$$F(v,w) = \varphi(v)(w) = H(w, \psi^{-1} \circ \varphi(v)) = H(w, \alpha(v)).$$

Replacing w by $\alpha(w)$ and using the symmetry of F gives

$$H(\alpha(w), \alpha(v)) = F(v, \alpha(w)) = F(\alpha(w), v) = H(v, \alpha^2(w)) = H(v, \lambda w).$$

Now choose $v = w$, $v \neq 0$. We get

$$H(\alpha(v), \alpha(v)) = H(v, \lambda v) = \bar{\lambda} H(v,v).$$

Since both $H(\alpha(v), \alpha(v))$ and $H(v, v)$ are real and positive, it follows that $\bar{\lambda}$ is real and positive, and thus the same is true for $\lambda = \bar{\lambda}$. $\square$

Returning to the proof of Theorem 2.7, define $\gamma = \lambda^{-1/2}\alpha$. Then γ is a conjugate linear isomorphism and $\gamma^2 = \mathrm{Id}$. Finally, γ commutes with the G-action, and so as in the above discussion on real structures, γ defines a realstructure on V for which ρ_V is a real representation.

Thus we have proved all of the statements in Theorem 2.7 except for the fact that, in case (iii), $\dim V$ is even. This is a general linear algebra fact about vector spaces for which there exists an $F \in \bigwedge^2 V^*$ such that the corresponding map $V \to V^*$ is an isomorphism. $\square$

3 A computational characterization

We would like a computational method for deciding when a representation can be defined over $\mathbb{R}$. First, a definition:

Definition 3.1. Let f be a class function on G. Define a new function $\psi_2(f)$ by

$$\psi_2(f)(g) = f(g^2).$$

Then $\psi_2(f)$ is also a class function, since

$$\psi_2(f)(xgx^{-1}) = f((xgx^{-1})^2) = f(xg^2x^{-1}) = f(g^2) = \psi_2(f)(g).$$

(One can define $\psi_n(f)$ similarly for every $n \in \mathbb{Z}$.)

In particular, for a character χ_V of G, we can consider the expression

$$\langle \psi_2(\chi_V), 1 \rangle = \frac{1}{\#(G)} \sum_{g \in G} \chi_V(g^2).$$

Theorem 3.2. *Let V be an irreducible G-representation.*

(i) *V and V^* are not isomorphic* $\iff$ $\langle \psi_2(\chi_V), 1 \rangle = 0$.

(ii) *$V \cong V^*$ and V is defined over $\mathbb{R}$* $\iff$ $\langle \psi_2(\chi_V), 1 \rangle = 1$.

(iii) *$V \cong V^*$ and V is not defined over $\mathbb{R}$* $\iff$ $\langle \psi_2(\chi_V), 1 \rangle = -1$.

Proof. By Theorem 2.7, we have the following:

(i) V and V^* are not isomorphic $\iff$ $\mathrm{Hom}^G(V, V^*) = \mathrm{Bil}^G(V) = 0$ $\iff$ $\langle \chi_{\mathrm{Bil}^G(V)}, 1 \rangle = 0$.

(ii) $V \cong V^*$ and V is defined over $\mathbb{R} \iff \dim(\mathrm{Sym}^2 V^*) = 1$ and $\bigwedge^2 V^* = 0 \iff \langle \chi_{\mathrm{Sym}^2 V^*}, 1\rangle = 1$ and $\langle \chi_{\bigwedge^2 V^*}, 1\rangle = 0$.

(iii) $V \cong V^*$ and V is not defined over $\mathbb{R} \iff \dim(\bigwedge^2 V^*) = 1$ and $\mathrm{Sym}^2 V^* = 0 \iff \langle \chi_{\bigwedge^2 V^*}, 1\rangle = 1$ and $\langle \chi_{\mathrm{Sym}^2 V^*}, 1\rangle = 0$.

So we must compute these characters. In fact, we claim:

(i)′ $\chi_{\mathrm{Bil}^G(V)} = \chi_{\mathrm{Hom}^G(V,V^*)} = \overline{\chi}_V^2$.

(ii)′ $\chi_{\mathrm{Sym}^2 V^*} = \frac{1}{2}(\overline{\chi}_V^2 + \psi_2(\overline{\chi}_V))$.

(iii)′ $\chi_{\bigwedge^2 V^*} = \frac{1}{2}(\overline{\chi}_V^2 - \psi_2(\overline{\chi}_V))$.

Assuming this, we have

$$\dim(\mathrm{Sym}^2 V^*) = \langle \chi_{\mathrm{Sym}^2 V^*}, 1\rangle = \frac{1}{2}(\langle \overline{\chi}_V^2, 1\rangle + \langle \psi_2(\overline{\chi}_V), 1\rangle);$$
$$\dim(\bigwedge^2 V^*) = \langle \chi_{\bigwedge^2 V^*}, 1\rangle = \frac{1}{2}(\langle \overline{\chi}_V^2, 1\rangle - \langle \psi_2(\overline{\chi}_V), 1\rangle).$$

Then $\langle \psi_2(\overline{\chi}_V), 1\rangle = 0 \iff \dim(\mathrm{Sym}^2 V^*) = \dim(\bigwedge^2 V^*)$, which happens exactly when $\mathrm{Hom}^G(V, V^*) = 0$, since otherwise one of the dimensions is 0 and the other is 1. Since $\langle \psi_2(\overline{\chi}_V), 1\rangle$ is real in this case,

$$\langle \psi_2(\overline{\chi}_V), 1\rangle = \langle \psi_2(\chi_V), 1\rangle.$$

A brief computation in the remaining cases shows that $\langle \chi_{\mathrm{Sym}^2 V^*}, 1\rangle = 1$ and $\langle \chi_{\bigwedge^2 V^*}, 1\rangle = 0 \iff \langle \psi_2(\overline{\chi}_V), 1\rangle = 1$, and $\langle \chi_{\bigwedge^2 V^*}, 1\rangle = 1$ and $\langle \chi_{\mathrm{Sym}^2 V^*}, 1\rangle = 0 \iff \langle \psi_2(\overline{\chi}_V), 1\rangle = -1$. As before, by taking conjugates, the first case happens $\iff \langle \psi_2(\chi_V), 1\rangle = 1$ and the second $\iff \langle \psi_2(\chi_V), 1\rangle = -1$.

So we must prove the claim. For $g \in G$, the linear map $\rho_V(g)$ is diagonalizable. Let $v_1, \dots, v_n$ be a basis of V such that $\rho_V(v_i) = \lambda_i v_i$ and let v_i^* be the dual basis. Then $\rho_V(g^2)$ is also diagonalized by the basis $v_1, \dots, v_n$, with eigenvalues λ_i^2, and hence

$$\psi_2(\overline{\chi}_V)(g) = \overline{\chi}_V(g^2) = \sum_i \bar{\lambda}_i^2.$$

Let $v_i^* v_j^* \in \mathrm{Hom}(V, V^*)$ be the linear map defined by $v_i^* v_j^*(w) = v_i^*(w) v_j^*$. Then $v_i^* v_j^*$, $1 \le i, j \le n$ is a basis for $\mathrm{Hom}(V, V^*)$ and each $v_i^* v_j^*$ is an eigenvector for $\rho_{\mathrm{Hom}(V,V^*)}(g)$ with eigenvalue $\lambda_i^{-1}\lambda_j^{-1} = \bar{\lambda}_i\bar{\lambda}_j$. Thus we see as previously noted that

$$\chi_{\mathrm{Hom}(V,V^*)}(g) = \sum_{i,j} \bar{\lambda}_i\bar{\lambda}_j = \left(\sum_i \bar{\lambda}_i\right)\left(\sum_j \bar{\lambda}_j\right) = (\overline{\chi}_V(g))^2.$$

We can also write this as

$$(\overline{\chi}_V(g))^2 = \sum_i \bar{\lambda}_i^2 + 2\sum_{i<j} \bar{\lambda}_i\bar{\lambda}_j.$$

As for $\mathrm{Sym}^2 V^*$, we can find a basis for it by symmetrizing the expressions $v_i^* v_j^*$ to $\frac{1}{2}(v_i^* v_j^* + v_j^* v_i^*)$. This expression is unchanged by switching i and i, and the functions

$$\frac{1}{2}(v_i^* v_j^* + v_j^* v_i^*), \qquad i \le j$$

are linearly independent. A similar argument shows that a basis for $\bigwedge^2 V^*$ is given by

$$\frac{1}{2}(v_i^* v_j^* - v_j^* v_i^*), \qquad i < j.$$

Thus

$$\chi_{\mathrm{Sym}^2 V^*}(g) = \sum_{i\le j} \bar{\lambda}_i\bar{\lambda}_j;$$
$$\chi_{\bigwedge^2 V^*}(g) = \sum_{i<j} \bar{\lambda}_i\bar{\lambda}_j.$$

Then

$$\begin{aligned}\frac{1}{2}(\overline{\chi}_V^2(g) + \psi_2(\overline{\chi}_V)(g)) &= \frac{1}{2}\left(\sum_i \bar{\lambda}_i^2 + 2\sum_{i<j} \bar{\lambda}_i\bar{\lambda}_j + \sum_i \bar{\lambda}_i^2\right)\\ &= \frac{1}{2}\left(2\sum_i \bar{\lambda}_i^2 + 2\sum_{i<j} \bar{\lambda}_i\bar{\lambda}_j\right)\\ &= \sum_{i\le j} \bar{\lambda}_i\bar{\lambda}_j = \chi_{\mathrm{Sym}^2 V^*}(g).\end{aligned}$$

A similar calculation establishes the formula for $\chi_{\bigwedge^2 V^*}(g)$. $\square$

Example 3.3. (1) Let $G = D_4$ and let V be the irreducible 2-dimensional representation of D_4. The elements of D_4 are α^k, $0 \le k \le 3$, and $\tau\alpha^k$, $0 \le k \le 3$. Moreover, $(\alpha^k)^2 = 1$ if $k = 0, 2$, $(\alpha^k)^2 = \alpha^2$ if $k = 1, 3$, and $(\tau\alpha^k)^2 = 1$ for all k. Thus, in D_4, there are 6 elements whose square is 1 and 2 elements whose square is α^2. We know that $\chi_V(1) = 2$ and that $\chi_V(\alpha^2) = -2$. Then

$$\langle \psi_2(\chi_V), 1\rangle = \frac{1}{8}(6 \cdot 2 + 2 \cdot (-2)) = 1.$$

Thus the irreducible 2-dimensional representation of D_4 can be defined over $\mathbb{R}$. Of course, we have seen this directly.

(2) Let $G = Q$, the quaternion group, and let V be the irreducible 2-dimensional representation of Q. The elements of Q are $\pm 1, \pm i, \pm j, \pm k$. Moreover, $(1)^2 = (-1)^2 = 1$ and all other elements have square -1. Thus, in Q, there are 2 elements whose square is 1 and 6 elements whose square is -1. We know that $\chi_V(1) = 2$ and that $\chi_V(-1) = -2$. Then

$$\langle \psi_2(\chi_V), 1\rangle = \frac{1}{8}(2 \cdot 2 + 6 \cdot (-2)) = -1.$$

Thus the irreducible 2-dimensional representation of Q **cannot** be defined over $\mathbb{R}$.

4 Irreducible real representations

In this section, we switch gears and look at things from the perspective of an irreducible real representation $V_{\mathbb{R}}$. We shall just state the main result (although its proof is not that difficult).

Recall that, if $V_{\mathbb{R}}$ is a real representation which is irreducible as a real representation, then Schur's lemma only says that $\mathrm{Hom}^G(V_{\mathbb{R}}, V_{\mathbb{R}})$ is a division ring containing $\mathbb{R}$ in its center, and is a finite dimensional $\mathbb{R}$-vector space since it is isomorphic to a vector subspace of $\mathbb{M}_n(\mathbb{R})$ for $n = \dim V_{\mathbb{R}}$. It is not hard to classify such division rings: $\mathrm{Hom}^G(V_{\mathbb{R}}, V_{\mathbb{R}})$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, the quaternions. We can then look at the complexification V of $V_{\mathbb{R}}$. Although $V_{\mathbb{R}}$ is an irreducible real representation, V need not be irreducible. The possibilities are as follows:

Theorem 4.1. *Let $V_{\mathbb{R}}$ be an irreducible real representation and let V be its complexification.*

(i) $\mathrm{Hom}^G(V_{\mathbb{R}}, V_{\mathbb{R}}) \cong \mathbb{R} \iff V$ *is irreducible.*

(ii) $\mathrm{Hom}^G(V_{\mathbb{R}}, V_{\mathbb{R}}) \cong \mathbb{C} \iff V \cong W \oplus W^*$, *where* W *and* W^* *are irreducible and* W *and* W^* *are not isomorphic.*

(iii) $\mathrm{Hom}^G(V_{\mathbb{R}}, V_{\mathbb{R}}) \cong \mathbb{H} \iff V \cong W \oplus W$, *where* W *is irreducible and* $W \cong W^*$. □

5 Real conjugacy classes

One question related to our previous discussion is the following: given a representation V of G, when is $\chi_V(g) \in \mathbb{R}$? Of course, if g is conjugate to g^{-1} then, for *every* representation V,

$$\chi_V(g) = \chi_V(g^{-1}) = \overline{\chi}_V(g),$$

and thus $\chi_V(g) \in \mathbb{R}$ for every V.

We make the following preliminary observation:

Lemma 5.1. *Let* $x \in G$, *and suppose that* y *is conjugate to* x. *Then* y^{-1} *is conjugate to* x^{-1}. *Hence,* $C(x)$ *is the conjugacy class of* x *and if we define*

$$C(x)^{-1} = \{y^{-1} : y \in C(x)\},$$

then $C(x)^{-1} = C(x^{-1})$.

Proof. For the first statement, if $y = gxg^{-1}$, then

$$y^{-1} = (gxg^{-1})^{-1} = gx^{-1}g^{-1}.$$

Thus, if $y \in C(x)$, then $y^{-1} \in C(x^{-1})$ and so $C(x)^{-1} \subseteq C(x^{-1})$. Conversely, if $z \in C(x^{-1})$, then z is conjugate to x^{-1} and hence $y = z^{-1}$ is conjugate to $(x^{-1})^{-1} = x$. Then by definition $z = y^{-1} \in C(x)^{-1}$, so that $C(x^{-1}) \subseteq C(x)^{-1}$. Thus $C(x)^{-1} = C(x^{-1})$. □

Definition 5.2. A conjugacy class $C(x)$ is *real* if $C(x)^{-1} = C(x)$, or equivalently if there exists a $y \in C(x)$ such that y is conjugate to y^{-1}. By the lemma, if there exists one such y, then y is conjugate to y^{-1} for every $y \in C(x)$.

Example 5.3. (1) Clearly, $C(1) = \{1\}$ is a real conjugacy class.

(2) If G is abelian, then $C(x) = \{x\}$ for every $x \in G$. Thus $C(x)$ is a real conjugacy class $\iff x = x^{-1} \iff x$ has order 1 or 2. In particular, if G is an abelian group of odd order, then the only real conjugacy class is $C(1)$.

(3) In S_n, every element σ is conjugate to σ^{-1}, and thus every conjugacy class is real. In fact, every element σ can be written as $\sigma = \gamma_1 \cdots \gamma_\ell$, where each γ_i is a cycle of some length $n_i > 1$ and the γ_i are pairwise disjoint. As disjoint cycles commute,

$$\sigma^{-1} = (\gamma_1 \cdots \gamma_\ell)^{-1} = \gamma_\ell^{-1} \cdots \gamma_1^{-1} = \gamma_1^{-1} \cdots \gamma_\ell^{-1}.$$

But each γ_i^{-1} is also a cycle of length n_i, and it is easy to check that, in S_n, two elements $\gamma_1 \cdots \gamma_\ell$ and $\delta_1 \cdots \delta_\ell$, both products of disjoint cycles of the same lengths, are conjugate.

(4) It is easy to check that the quaternion group Q also has the property that every element g is conjugate to g^{-1}, and thus that every conjugacy class is real.

As usual, enumerate the irreducible representations of G up to isomorphism as $V_1, \dots, V_h$. We then have the following curious fact about real conjugacy classes:

Theorem 5.4 (Burnside)**.** *The number of real conjugacy classes of G is equal to the number of i such that the irreducible representation V_i is isomorphic to V_i^*, or equivalently such that $\chi_{V_i}(g) \in \mathbb{R}$ for all $g \in G$.*

Proof. Enumerate the set of conjugacy classes of G as $C(x_1), \dots, C(x_h)$. Note that this enumeration doesn't necessarily have anything to do with the enumeration $V_1, \dots, V_h$ of irreducible representations chosen above. As usual, we let $Z \subseteq L^2(G)$ be the subspace of class functions. Then there are two natural bases for Z: the set of characteristic functions $f_{C(x_i)}$ and the set of characters χ_{V_i}. We abbreviate $f_{C(x_i)}$ by f_i. There are two permutations τ and σ of the index set $\{1, \dots, h\}$. We let $\tau(i)$ be the unique j such that $C(x_i)^{-1} = C(x_j)$, and we let $\sigma(i)$ be the unique j such that $V_i^* = V_j$. The content of the theorem is then that the number of i such that $\tau(i) = i$ is equal to the number of i such that $\sigma(i) = i$.

As permutations of the index set, both τ and σ define permutation matrices $P_\tau, P_\sigma \in GL(h, \mathbb{C})$ by the rule $P_\tau(f_i) = f_{\tau(i)}$, and similarly $P_\sigma(f_i) = f_{\sigma(i)}$. As with all permutation matrices, $\operatorname{Tr} P_\tau$ is the number of i such that $\tau(i) = i$ and $\operatorname{Tr} P_\sigma$ is the number of i such that $\sigma(i) = i$. So we must show that $\operatorname{Tr} P_\tau = \operatorname{Tr} P_\sigma$. It suffices to find an invertible $h \times h$ matrix M such that $P\tau \cdot M = M \cdot P_\sigma$, for then $P_\tau = M P_\sigma M^{-1}$ and so the traces are equal.

Let $M = (\chi_{V_j}(x_i))$. Then M is the change of basis matrix for the two bases $f_1, \dots, f_h$ and $\chi_{V_1}, \dots, \chi_{V_h}$, because

$$M f_i = \sum_{j=1}^{h} \chi_{V_i}(x_j) f_j = \sum_{j=1}^{h} \chi_{V_i}(x_j) f_{C(x_j)},$$

and by comparing the values of the above class function on every x_j we see that $Mf_i = \chi_{V_i}$. In particular, M is invertible. By definition, since P_τ acts by permuting the f_i according to τ, we see that

$$\begin{aligned} P_\tau M f_i &= \sum_{j=1}^{h} \chi_{V_i}(x_j) f_{\tau(j)} = \sum_{j=1}^{h} \chi_{V_i}(x_j) f_{C(x_j^{-1})} \\ &= \sum_{j=1}^{h} \chi_{V_i}(x_j^{-1}) f_{C(x_j)} = \sum_{j=1}^{h} \chi_{V_i^*}(x_j) f_{C(x_j)} \\ &= \chi_{V_i^*}. \end{aligned}$$

On the other hand,

$$MP_\sigma f_i = Mf_{\sigma(i)} = \chi_{V_{\sigma(i)}} = \chi_{V_i^*}.$$

Thus $P\tau \cdot M = M \cdot P_\sigma$, concluding the proof. □

The following is a purely group-theoretic argument:

Proposition 5.5. *The order $\#(G)$ is odd $\iff$ the only real conjugacy class is $C(1) = \{1\}$.*

Proof. Equivalently, we have to show that the order $\#(G)$ is even $\iff$ there exists a real conjugacy class $C(x)$ with $x \neq 1$. If $\#(G)$ is even, then an easy special case of Cauchy's theorem says that there exists an element x of order 2. Then $x \neq 1$ and $x = x^{-1}$, so that $C(x)$ is a real conjugacy class and $C(x) \neq C(1)$.

Conversely, suppose that there exists a real conjugacy class $C(x)$ with $x \neq 1$. If $x = x^{-1}$, then x has order 2. By Lagrange's theorem, the order of any element of G divides the order of G, so $\#(G)$ is even in this case. Otherwise, $x \neq x^{-1}$ but there exists an $h \in G$ such that $hxh^{-1} = x^{-1}$. Then $h^2xh^{-2} = h(hxh^{-1})h^{-1} = hx^{-1}h^{-1} = x$, and by induction we see that

$$h^a x h^{-a} = \begin{cases} x^{-1}, & \text{if } a \text{ is odd;} \\ x, & \text{if } a \text{ is even.} \end{cases}$$

Let N be the order of h. Then N must be even, since $h^N x h^{-N} = 1x1 = x$ and $x \neq x^{-1}$. But then N divides $\#(G)$, again by Lagrange's theorem, so that $\#(G)$ is divisible by an even number and hence is even. □

Corollary 5.6. *The order $\#(G)$ is odd $\iff$ the only irreducible representation V_i such that $V_i \cong V_i^*$, or equivalently such that $\chi_{V_i}(g) \in \mathbb{R}$ for all $g \in G$, is the trivial representation.*

Proof. By the previous corollary, $\#(G)$ is odd $\iff$ there exists exactly one real conjugacy class $\iff$ there exists exactly one irreducible representation V of G up to isomorphism such that $V \cong V^*$. Since the trivial representation has this property, if there is only one such it must be the trivial representation. $\square$

We then have the following purely group-theoretic fact:

Proposition 5.7. *If $\#(G)$ is odd and the number of conjugacy classes of G is h, then*

$$h \equiv \#(G) \pmod{16}.$$

Proof. We use the following basic fact: if n is an odd integer, then $n^2 \equiv 1 \pmod 8$. This can be proved by checking all the possibilities (n must be $\equiv 1, 3, 5, 7 \pmod 8$), or directly: $n = 2m + 1$ for some integer m, so that

$$n^2 = (2m+1)^2 = 4m^2 + 4m + 1 = 4m(m+1) + 1.$$

But $m(m+1)$ is always even, so $n^2 \equiv 1 \pmod 8$.

Now let h be the number of conjugacy classes of G or equivalently the number of irreducible representations up to isomorphism. If we enumerate these as $V_1, \ldots, V_h$, we can assume that V_1 is the trivial representation. If $d_i = \dim V_i$, then $d_1 = 1$ and d_i divides $\#(G)$, hence $d_i = 2e_i + 1$. Moreover, if $i \neq 1$, then V_i and V_i^* are not isomorphic, so there are $2r$ remaining representations V_i and they occur in pairs V_i, V_i^* with $\dim V_i^* = \dim V_i = 2e_i + 1$. For example, we could index the representations so that V_1 is the trivial representation and $V_{i+2} \cong V_2^*, \ldots, V_{2r+1} \cong V_{r+1}^*$. Hence $d_i = d_{i+r}$ for $2 \leq i \leq r+1$. Then

$$\begin{aligned}
\#(G) = \sum_{i=1}^{h} d_i^2 &= 1 + \sum_{i=2}^{2r+1} d_i^2 = 1 + 2\sum_{i=2}^{r+1} d_i^2 \\
&= 1 + 2\sum_{i=2}^{r+1} (2e_i+1)^2 = 1 + \sum_{i=2}^{r+1} 2(4e_i^2 + 4e_i + 1) \\
&= 1 + \sum_{i=2}^{r+1} 8e_i(e_i+1) + 2r = 2r + 1 + \sum_{i=2}^{r+1} 8e_i(e_i+1).
\end{aligned}$$

As before, $8e_i(e_i+1) \equiv 0 \pmod{16}$, so that

$$\#(G) \equiv 2r + 1 = h \pmod{16}$$

as claimed. $\square$

6 The case of the rational numbers

We begin with some general comments about the possible values of a character χ_V of a finite group G.

Definition 6.1. If G is a finite group, then the *exponent* of G is the least common multiple of the orders of the elements of G. Equivalently, N is the smallest positive integer such that $g^N = 1$ for all $g \in G$. Note that N divides $\#(G)$ and that (by Cauchy's theorem) N and $\#(G)$ have the same prime factors. However, N can be strictly smaller that $\#(G)$. For example, for D_4, $N = 4$ but $\#(D_4) = 8$. More generally, if $\#(G) = p^n$ where p is a prime, then the exponent of G is $p^n \iff G$ is cyclic. For another example, the exponent of S_4 is 12 but $\#(S_4) = 24$.

Suppose now that ρ_V is a G-representation. For every $g \in G$, the eigenvalues λ_i of $\rho_V(g)$ are a^{th} roots of unity, where a is the order of g, and hence they are N^{th} roots of unity, where N is the exponent of G. Since $\chi_V(g)$ is the sum of the λ_i, it follows that, for every $g \in G$, $\chi_V(g) \in \mathbb{Q}(\mu_N)$, the extension of $\mathbb{Q}$ generated by the N^{th} roots of unity.

We can then ask when a G-representation V is defined over $\mathbb{Q}$. In general, this is a very hard question. An easy question to ask is: given a finite group G, when is $\chi_V(g) \in \mathbb{Q}$ for every representation V of G (or equivalently every irreducible representation) and every $g \in G$? Note that $\chi_V(g) \in \mathbb{Q}$ $\iff$ $\chi_V(g) \in \mathbb{Z}$, since $\chi_V(g)$ is an algebraic integer. This question can be answered:

Theorem 6.2. *For a finite group G, $\chi_V(g) \in \mathbb{Q}$ for every representation V of G and every element $g \in G \iff$ for every $g \in G$ and every $a \in \mathbb{Z}$ such that $\gcd(a, \#(G)) = 1$, g^a is conjugate to g.*

Proof. We shall give the proof modulo a little Galois theory and number theory. Note that $\mathbb{Q}(\mu_N)$ is a normal, hence Galois extension of $\mathbb{Q}$ since it is the splitting field of $x^N - 1$. Thus, given $\alpha \in \mathbb{Q}(\mu_N)$, $\alpha \in \mathbb{Q} \iff \sigma(\alpha) = \alpha$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$. Moreover, $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^*$. In fact, let ζ be a generator of the cyclic group μ_N. For example, we could take $\zeta = e^{2\pi i/N}$. Then for all $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$, $\sigma(\zeta)$ is another generator of μ_N, hence $\sigma(\zeta) = \zeta^a$ for some integer a mod N, necessarily relatively prime to N. Viewing a as an element of $(\mathbb{Z}/N\mathbb{Z})^*$, the map $\sigma \mapsto a$ sets up an isomorphism $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^*$. Note also that, if $\sigma(\zeta) = \zeta^a$, then $\sigma(\lambda) = \lambda^a$ for all $\lambda \in \mu_N$.

If V is a G-representation, then $\chi_V(g) = \sum_i \lambda_i$, where the $\lambda_i \in \mathbb{Q}(\mu_N)$. Moreover, given $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ corresponding to $a \in (\mathbb{Z}/N\mathbb{Z})^*$,

$$\sigma(\chi_V(g)) = \sum_i \sigma(\lambda_i) = \sum_i \lambda_i^a = \chi_V(g^a).$$

Thus $\chi_V(g) \in \mathbb{Q}$ for every representation V of G and every element $g \in G$ $\iff$ $\chi_V(g) = \chi_V(g^a)$ for every representation V of G, every element $g \in G$, and every $a \in \mathbb{Z}$ which is relatively prime to N. Since the functions χ_V span the space of class functions, this is the case $\iff$ g is conjugate to g^a for every $g \in G$ and every $a \in \mathbb{Z}$ which is relatively prime to N. Finally, since N and $\#(G)$ have the same prime factors, a is relatively prime to N $\iff$ a is relatively prime to $\#(G)$. $\square$

Example 6.3. (1) The symmetric group S_n has the property that, for every $\sigma \in S_n$ and every $a \in \mathbb{Z}$ which is relatively prime to $\#(S_n) = n!$, σ^a is conjugate to σ. In fact, we have seen that $\sigma = \gamma_1 \cdots \gamma_\ell$, where the γ_i are pairwise disjoint cycles of lengths $n_i > 1$. Since the γ_i commute,

$$\sigma^a = (\gamma_1 \cdots \gamma_\ell)^a = \gamma_1^a \cdots \gamma_\ell^a.$$

As each γ_i is a cycle of length n_i and $\gcd(a, n_i) = 1$ since n_i divides $n!$, a Modern Algebra I argument shows that γ_i^a is an n_i-cycle for every i. Also, the elements of $\{1, \dots, n\}$ appearing in γ_i^a are the same as the elements appearing in γ_i, so that $\gamma_1^a, \dots, \gamma_\ell^a$ are disjoint cycles of lengths n_i. As we have seen before, this implies that $\sigma^a = \gamma_1^a \cdots \gamma_\ell^a$ is conjugate to σ. We shall see that, in fact, every representation of S_n can be defined over $\mathbb{Q}$.

(2) For the quaternion group Q, as $\#(Q) = 8$, $a \in \mathbb{Z}$ is relatively prime to $\#(Q)$ $\iff$ a is odd. For odd a, it is easy to check that g^a is conjugate to g for every $g \in Q$. For example, $1^a = 1$, $(-1)^a = -1$, and $(\pm i)^a = \pm i$ if $a \equiv 1 \pmod 4$ and $(\pm i)^a = \mp i$ if $a \equiv 3 \pmod 4$. But i and $-i$ are conjugate, since e.g. $-i = jij^{-1}$. Thus (as is easy to see directly) $\chi_V(g) \in \mathbb{Q}$ for every representation V of Q and every element $g \in Q$. On the other hand, the irreducible 2-dimensional representation of Q cannot be defined over $\mathbb{Q}$. In fact, we have seen that it cannot be defined over $\mathbb{R}$.