LECTURE NOTES A

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1. Theorem of this lecture

Let k be a field. Let X be a proper scheme over k. We say a pair (ω_X, t) is a *dualizing sheaf* or *dualizing module* for X if ω_X is a coherent $\mathcal{O}_X$-module and

$$t : H^{\dim X}(X, \omega_X) \longrightarrow k$$

is a k-linear map such that the pair (ω_X, k) represents the functor

$$Coh(\mathcal{O}_X) \longrightarrow \mathrm{Vect}_k, \quad \mathcal{F} \longmapsto \mathrm{Hom}_k(H^{\dim X}(X, \mathcal{F}), k)$$

on the category of coherent $\mathcal{O}_X$-modules. Explicitly this says that for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ the map

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X) \times H^{\dim X}(X, \mathcal{F}) \longrightarrow k, \quad (\varphi, \xi) \longmapsto t(\varphi(\xi))$$

is a perfect pairing of finite dimensional k-vector spaces. The notation makes sense: since $\varphi : \mathcal{F} \to \omega_X$ is a map of $\mathcal{O}_X$-modules, we obtain an induced map $\varphi : H^n(X, \mathcal{F}) \to H^n(X, \omega_X)$ and we can apply this to the cohomology classe ξ whereupon we can use t to get an element of k.

Theorem 1.1. *If X is projective over k then there exists a dualizing sheaf. In fact, for any closed immersion $i : X \to P = \mathbf{P}^n_k$ there is an isomorphism*

$$i_*\omega_X = \mathcal{E}xt^{n-\dim X}_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P)$$

In this lecture we will try to indicate the proof of this theorem and compute what happens in a special case.

2. Preliminaries on Ext

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Recall that $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, -)$ are the right derived functors of the sheaf-hom functor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$. Similarly, $\mathrm{Ext}^p_X(\mathcal{F}, -)$ are the right derived functors of the functor $\mathrm{Hom}_X(\mathcal{F}, -)$ of global homomorphisms of $\mathcal{O}_X$-modules.

Remark 2.1. On any ringed space $(X, \mathcal{O}_X)$ the formation of $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ commutes with restriction to opens. This is clear from the fact that an injective resolution of $\mathcal{G}$ restricts to an injective resolution of $\mathcal{G}$ on any open and that the formation of $\mathcal{H}om$ commutes with restriction to opens.

Remark 2.2. For any short exact sequence $0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0$ of $\mathcal{O}_X$-modules we obtain a long exact sequence

$$0 \to \mathcal{H}om(\mathcal{F}, \mathcal{G}_1) \to \mathcal{H}om(\mathcal{F}, \mathcal{G}_2) \to \mathcal{H}om(\mathcal{F}, \mathcal{G}_3) \to \mathcal{E}xt^1(\mathcal{F}, \mathcal{G}_1) \to \mathcal{E}xt^1(\mathcal{F}, \mathcal{G}_2) \to \ldots$$

(we are dropping the subscript $\mathcal{O}_X$ here in order to fit this onto one line in the pdf). This is a general fact about derived functors. For any short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of $\mathcal{O}_X$-modules and an $\mathcal{O}_X$-module $\mathcal{G}$ we obtain a long exact sequence

$$0 \to \mathcal{H}om(\mathcal{F}_3, \mathcal{G}) \to \mathcal{H}om(\mathcal{F}_2, \mathcal{G}) \to \mathcal{H}om(\mathcal{F}_1, \mathcal{G}) \to \mathcal{E}xt^1(\mathcal{F}_3, \mathcal{G}) \to \mathcal{E}xt^1(\mathcal{F}_2, \mathcal{G}) \to \ldots$$

This follows by choosing an injective resolution of $\mathcal{G}$ and arguing exactly as in the case of modules over rings.

Remark 2.3. For any short exact sequence $0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0$ of $\mathcal{O}_X$-modules we obtain a long exact sequence

$$0 \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}_1) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}_2) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}_3) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{G}_1) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{G}_2) \to \ldots$$

(we are dropping the subscript X here in order to fit this onto one line in the pdf). This is a general fact about derived functors. For any short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of $\mathcal{O}_X$-modules and an $\mathcal{O}_X$-module $\mathcal{G}$ we obtain a long exact sequence

$$0 \to \mathrm{Hom}(\mathcal{F}_3, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}_2, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}_1, \mathcal{G}) \to \mathrm{Ext}^1(\mathcal{F}_3, \mathcal{G}) \to \mathrm{Ext}^1(\mathcal{F}_2, \mathcal{G}) \to \ldots$$

This follows by choosing an injective resolution of $\mathcal{G}$ and arguing exactly as in the case of modules over rings.

Lemma 2.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. For any finite locally free module $\mathcal{F}$ we have $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = 0$ for $p > 0$ and any $\mathcal{O}_X$-module $\mathcal{G}$.*

Proof. We may work locally on X. Hence we may assume $\mathcal{F} = \mathcal{O}_X^{\oplus n}$. To see the claim is true, we observe that

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, \mathcal{H}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{H})^{\oplus n} = \mathcal{H}^{\oplus n}$$

is an exact functor in the $\mathcal{O}_X$-module $\mathcal{H}$ and hence has vanishing higher derived functors. $\square$

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a finite locally free module. We set

$$\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

and we call it the *dual* finite locally free module. For any $\mathcal{O}_X$-module $\mathcal{G}$ the canonical evaluation map

$$\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

is an isomorphism of $\mathcal{O}_X$-modules.

Lemma 2.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. For any finite locally free module $\mathcal{F}$ we have $\mathrm{Ext}^p_X(\mathcal{F}, \mathcal{G}) = H^p(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{G})$ for any $\mathcal{O}_X$-module $\mathcal{G}$. Here $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is the dual finite locally free module.*

Proof. Discussed in a previous lecture. Hint: the functor $\mathrm{Hom}_X(\mathcal{F}, -)$ is equal to the functor $H^0(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} -)$ by the discussion above and then take higher derived functors on both sides. $\square$

Lemma 2.6. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

(1) *the sheaves* $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ *are quasi-coherent,*
(2) *if* $\mathcal{G}$ *is coherent as well, then* $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ *is coherent, and*
(3) *if* $X = \mathrm{Spec}(A)$ *and* $\mathcal{F}$ *and* $\mathcal{G}$ *correspond to the* A*-modules* M *and* N*, then we have* $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \widetilde{\mathrm{Ext}^p_A(M, N)}$ *on* X.

Proof. Parts (1) and (2) are local on X. Hence it suffices to prove part (3) because we already know that $\mathrm{Ext}^p_A(M, N)$ is a finite A-module if M and N are finite modules over a Noetherian ring A, see Lemma 08YR.

Proof of part (3). We will prove this by induction on p. If $p = 0$, then we have to show that

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \widetilde{\mathrm{Hom}_A(M, N)}$$

on X. This follows by evaluating both sides on $D(f) = \mathrm{Spec}(A_f)$ for $f \in A$. For $p > 0$ choose a short exact sequence

$$0 \to M' \to A^{\oplus n} \to M \to 0$$

which leads to a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{O}_X^{\oplus n} \to \mathcal{F} \to 0$$

since $\mathcal{F} = \widetilde{M}$. By Lemma 2.4 we have $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, \mathcal{G}) = 0$ for $p > 0$. Using the long exact sequences for $\mathcal{E}xt$ (see remark above), we obtain an exact sequence

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}', \mathcal{G}) \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to 0$$

and isomorphisms $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}', \mathcal{G}) \to \mathcal{E}xt^{p+1}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ for all $p \geq 1$. Since we have similar results for Hom_A and Ext_A we conclude what we want. □

3. Trivial duality

Let $i : X \to P$ be a closed immersion of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{G}$ be an $\mathcal{O}_P$-module. Then we have the equalities

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_P}(i_*\mathcal{F}, \mathcal{H}om_{\mathcal{O}_P}(i_*\mathcal{O}_X, \mathcal{G})) &= \mathrm{Hom}_{i_*\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{H}om_{\mathcal{O}_P}(i_*\mathcal{O}_X, \mathcal{G})) \\ &= \mathrm{Hom}_{\mathcal{O}_P}(i_*\mathcal{F}, \mathcal{G}) \end{aligned}$$

The first equality is true because both $i_*\mathcal{F}$ and $\mathcal{H}om_{\mathcal{O}_P}(i_*\mathcal{O}_X, \mathcal{G})$ are annihilated by the kernel of the surjection $\mathcal{O}_P \to i_*\mathcal{O}_X$. The second equality is a special case of the very general Lemma 0A6F. In fact, this lemma shows that the functor

$$Mod(\mathcal{O}_P) \longrightarrow Mod(i_*\mathcal{O}_X), \quad \mathcal{G} \longmapsto \mathcal{H}om_{\mathcal{O}_P}(i_*\mathcal{O}_X, \mathcal{G})$$

is the right adjoint to the exact functor $Mod(i_*\mathcal{O}_X) \to Mod(\mathcal{O}_P)$. Hence by the already discussed Lemma 015Z if $\mathcal{I}$ is an injective $\mathcal{O}_P$-module, then $\mathcal{H}om_{\mathcal{O}_P}(i_*\mathcal{O}_X, \mathcal{I})$ is an injective $i_*\mathcal{O}_X$-module.

Lemma 3.1. *Let* $\mathcal{A}$ *be an abelian category. Let* $I^\bullet$ *be a bounded below complex of injective objects of* $\mathcal{A}$*. Let* c *be the smallest index such that* $H^c(I^\bullet)$ *is nonzero. Then for any* A *in* $\mathcal{A}$ *the complex* $\mathrm{Hom}(A, I^\bullet)$ *is acyclic in degrees* $< c$ *and* $H^c(\mathrm{Hom}(A, I^\bullet)) = \mathrm{Hom}(A, H^c(I^\bullet))$.

Proof. Good exercise. □

4. Proof of the theorem

See Hartshorne proof of Proposition 7.5 in chapter III.

Choose a closed immersion $i : X \to P = \mathbf{P}^n_k$ as in the statement of the theorem. Let ω_X be defined by the formula in the statement of the theorem; this makes sense by the . The theorem follows from the following string of equalities

$$\begin{aligned}
\mathrm{Hom}_X(\mathcal{F}, \omega_X) &= \mathrm{Hom}_P(i_*\mathcal{F}, i_*\omega_X) \\
&= \mathrm{Hom}_P(i_*\mathcal{F}, \mathcal{E}xt^{dim P - \dim X}_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P)) \\
&= \mathrm{Ext}^{\dim P - \dim X}_P(i_*\mathcal{F}, \omega_P) \\
&= \mathrm{Hom}_k(H^{\dim X}(P, i_*\mathcal{F}), k) \\
&= \mathrm{Hom}_k(H^{\dim X}(X, \mathcal{F}), k)
\end{aligned}$$

The first equality follows from the discussion in the last lecture. The second equality is our choice of ω_X. The third equality: see below. The fourth equality is duality on P we already proved. The final equality we saw before: cohomology of $\mathcal{F}$ on X and on the pushforward of $\mathcal{F}$ to P are the same.

Lemma 4.1. *In the situation above we have $\mathcal{E}xt^p_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P) = 0$ for $p < \dim P - \dim X$.*

Proof. By Lemma 2.6 and looking on the affine opens $D_+(T_i)$ of $P = \mathbf{P}^n_k$ this translated into the following algebra fact: Let $B = k[x_1, \ldots, x_n] \to A$ be a surjection with kernel I, then $\mathrm{Ext}^p_B(A, B) = \mathrm{Ext}^p(B/I, B) = 0$ for $p < n - \dim(A)$. To prove this, it suffices to show that $\mathrm{depth}_I(B) \geq n - \dim(A)$, see Lemma 0AVZ. The inequality $\mathrm{depth}_I(B) \geq n - \dim(A)$ is an immediate consequence of Lemma 0BUX. $\square$

Proof of third equality. Choose an injective resolution $\omega_P \to \mathcal{I}^\bullet$. By Lemma 4.1 and the definition of $\mathcal{E}xt$ the sheaf $\mathcal{E}xt^{\dim P - \dim X}_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P)$ is the first nonzero cohomology sheaf of the complex

$$\mathcal{H}om_{\mathcal{O}_P}(i_*\mathcal{O}_X, \mathcal{I}^\bullet)$$

Moreover, by Section 3 this is a complex of injective $i_*\mathcal{O}_X$-modules and we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_P}(i_*\mathcal{F}, \mathcal{H}om_{\mathcal{O}_P}(i_*\mathcal{O}_X, \mathcal{I}^\bullet)) &= \mathrm{Hom}_{i_*\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{H}om_{\mathcal{O}_P}(i_*\mathcal{O}_X, \mathcal{I}^\bullet)) \\
&= \mathrm{Hom}_{\mathcal{O}_P}(i_*\mathcal{F}, \mathcal{I}^\bullet)
\end{aligned}$$

The final complex computes $\mathrm{Ext}^\bullet_P(i_*\mathcal{F}, \omega_P)$ by definition. By Lemma 3.1 we obtain that the middle complex is acyclic in degrees $< \dim P - \dim X$ and equal to the left hand side of

$$\mathrm{Hom}_{i_*\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{E}xt^{\dim P - \dim X}_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P)) = \mathrm{Hom}_P(i_*\mathcal{F}, \mathcal{E}xt^{\dim P - \dim X}_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P))$$

in degree $\dim P - \dim X$; the equality holds because both the module $i_*\mathcal{F}$ and $\mathcal{E}xt^{\dim P - \dim X}_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P)$ are annihilated by the ideal sheaf $\mathrm{Ker}(\mathcal{O}_P \to i_*\mathcal{O}_X)$ of X in P. Thus we conclude that this is equal to $\mathrm{Ext}^{\dim P - \dim X}_P(i_*\mathcal{F}, \omega_P)$ as desired and the proof is complete[1].

[1] We also deduce that $\mathrm{Ext}^p_P(i_*\mathcal{F}, \omega_P) = 0$ for $p < \dim P - \dim X$, but this is irrelevant to the proof of the theorem.

5. Dualizing sheaf of a hypersurface

Suppose that $X \subset P = \mathbf{P}^n_k$ is a hypersurface. In other words, we have a nonzero homogeneous polynomial $F \in k[T_0, \ldots, T_n]$ of degree $d > 0$ such that

$$X = \mathrm{Proj}(k[T_0, \ldots, T_n]/(F))$$

as a closed subscheme of $P = \mathrm{Proj}(k[T_0, \ldots, T_n])$. Another way to say this is that on each of the standard affine opens $D_+(T_i) = \mathrm{Spec}(k[T_0/T_i, \ldots, T_n/T_i])$ we have that

$$X \cap D_+(T_i) = \mathrm{Spec}(k[T_0/T_i, \ldots, T_n/T_i]/(F(T_0/T_i, \ldots, T_n/T_i)))$$

The short exact sequence

$$0 \to k[T_0, \ldots, T_n](-d) \to k[T_0, \ldots, T_n] \to k[T_0, \ldots, T_n]/(F) \to 0$$

of graded modules gives rise (by the tilde functor) to a short exact sequence

$$0 \to \mathcal{O}(-d) \to \mathcal{O} \to i_*\mathcal{O}_X \to 0$$

of $\mathcal{O}_P$-modules. Here $i : X \to P$ denotes the given closed immersion. Applying the corresponding long exact sequence of $\mathcal{E}xt$ we obtain

$$0 \to \mathcal{H}om(i_*\mathcal{O}_X, \omega_P) \to \mathcal{H}om(\mathcal{O}, \omega_P) \to \mathcal{H}om(\mathcal{O}(-d), \omega_P) \to \mathcal{E}xt^1(i_*\mathcal{O}_X, \mathcal{G}) \to 0$$

because we have the vanishing $\mathcal{E}xt^1(\mathcal{O}, \omega_P)$ by Lemma 2.4. Using the fact that $\mathcal{O}$ and $\mathcal{O}(-d)$ are locally free we may rewrite this as

$$0 \to \mathcal{H}om(i_*\mathcal{O}_X, \omega_P) \to \omega_P \to \omega_P(d) \to \mathcal{E}xt^1(i_*\mathcal{O}_X, \omega_P) \to 0$$

An easy local calculation shows that the map $\omega_P \to \omega_P(d)$ in the middle is given by multiplication by F. What else could it be? On the other hand, tensoring the initial short exact sequence with $\omega_P(d)$ we obtain

$$0 \to \omega_P \to \omega_P(d) \to \omega_P(d) \otimes_{\mathcal{O}_P} i_*\mathcal{O}_X \to 0$$

By the projection formula, see Section 01E6, we have

$$\omega_P(d) \otimes_{\mathcal{O}_P} i_*\mathcal{O}_X = i^*(\omega_P(d))$$

Putting everything together we conclude

$$\mathcal{H}om(i_*\mathcal{O}_X, \omega_P) = 0$$

and

$$\omega_X = \mathcal{E}xt^1(i_*\mathcal{O}_X, \omega_P) = i^*(\omega_P(d)) = i^*(\mathcal{O}(d - n - 1)) = \mathcal{O}_X(d - n - 1)$$