Schemes

Definitions and results. Kähler differentials.

(a) Let $R \to A$ be a ring map. The module of Kähler differentials of A over R is

$$\Omega^1_{A/R} = \bigoplus\nolimits_{a \in A} A \cdot \mathrm{d}a \Big/ \langle \mathrm{d}(a_1 a_2) - a_1 \mathrm{d}a_2 - a_2 \mathrm{d}a_1, \mathrm{d}r \rangle.$$

The canonical universal R-derivation $\mathrm{d} : A \to \Omega^1_{A/R}$ maps $a \mapsto \mathrm{d}a$.

(b) Consider the short exact sequence

$$0 \to I \to A \otimes_R A \to A \to 0$$

which defines the ideal I. There is a canonical derivation $\mathrm{d} : A \to I/I^2$ which maps a to the class of $a \otimes 1 - 1 \otimes a$. This is another presentation of the module of derivations of A over R, in other words

$$(I/I^2, \mathrm{d}) \cong (\Omega^1_{A/R}, \mathrm{d}).$$

(c) For multiplicative subsets $S_R \subset R$ and $S_A \subset A$ such that S_R maps into S_A we have

$$\Omega^1_{S_A^{-1}A/S_R^{-1}R} = S_A^{-1}\Omega^1_{A/R}.$$

(d) If A is a finitely presented R-algebra then $\Omega^1_{A/R}$ is a finitely presented A-module. Hence in this case the *fitting* ideals of $\Omega^1_{A/R}$ are defined. (See exercise set 6 of last semester.)

(e) Let $f : X \to S$ be a morphism of schemes. There is a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\Omega^1_{X/S}$ and a $\mathcal{O}_S$-linear derivation

$$\mathrm{d} : \mathcal{O}_X \longrightarrow \Omega^1_{X/S}$$

such that for any affine opens $\operatorname{Spec} A \subset X$, $\operatorname{Spec} R \subset S$ with $f(\operatorname{Spec} A) \subset \operatorname{Spec} R$ we have

$$\Gamma(\operatorname{Spec} A, \Omega^1_{X/S}) = \Omega^1_{A/R}$$

compatibly with d.

1. Let $k[\epsilon]$ be the ring of dual numbers over the field k, i.e., $\epsilon^2 = 0$.

(a) Consider the ring map

$$R = k[\epsilon] \to A = k[x, \epsilon]/(\epsilon x)$$

Show that the fitting ideals of $\Omega^1_{A/R}$ are (starting with the zeroth fitting ideal)

$$(\epsilon), A, A, \ldots$$

(b) Consider the map $R = k[t] \to A = k[x, y, t]/(x(y-t)(y-1), x(x-t))$. Show that the fitting ideals of of $\Omega^1_{A/R}$ in A are (assume characteristic k is zero for simplicity)

$$x(2x-t)(2y-t-1)A,\ (x, y, t) \cap (x, y-1, t),\ A,\ A, \ldots$$

So the 0-the fitting ideal is cut out by a single element of A, the 1st fitting ideal defines two closed points of $\operatorname{Spec} A$, and the others are all trivial.

(c) Consider the map $R = k[t] \to A = k[x, y, t]/(xy - t^n)$. Compute the fitting ideals of $\Omega^1_{A/R}$.

Remark. The kth fitting ideal of $\Omega^1_{X/S}$ is commonly used to define the singular scheme of the morphism $X \to S$ when X has relative dimension k over S. But as part (a) shows, you have to be careful doing this when your family does not have "constant" fibre dimension, e.g., when it is not flat. As part (b) shows, flatness doesn't garantee it works either (and yes this is a flat family). In "good cases" – such as in (c) – for families of curves you expect the 0-th fitting ideal to be zero and the 1st fitting ideal to define (scheme-theoretically) the singular locus.

2. Suppose that R is a ring and

$$A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_n).$$

Note that we are assuming that A is presented by the same number of equations as variables. Thus the matrix of partial derivatives

$$(\partial f_i/\partial x_j)$$

is $n \times n$, i.e., a square matrix. Assume that its determinant is invertible as an element in A. Note that this is exactly the condition that says that $\Omega^1_{A/R} = (0)$ in this case of n-generators and n relations. Let $\pi : B' \to B$ be a surjection of R-algebras whose kernel J has square zero (as an ideal in B'). Let $\varphi : A \to B$ be a homomorphism of R-algebras. Show there exists a unique homomorphism of R-algebras $\varphi' : A \to B'$ such that $\varphi = \pi \circ \varphi'$.

3. Find a generalization of the result of the previous exercise to the case where $A = R[x, y]/(f)$.