STUDENT NOTE

1. Definitions

Definition 1.1. A ring map $A \to B$ is *finite* if B is finitely generated as an A-module.

Definition 1.2. Let $A \to B$ be a ring map. An element $x \in B$ is *integral* over A if there exists a monic polynomial $P(T) = T^d + a_1 T^{d-1} + \ldots + a_d \in A[T]$ such that $P(x) = 0$ in B.

Definition 1.3. A ring map $A \to B$ is *integral* if every element of B is integral over A.

Definition 1.4. Let $A \subset B$ be a ring extension. We say A is *integrally closed* in B if every element of B integral over A is contained in A.

Definition 1.5. A domain A is *normal* if A is integrally closed in its fraction field.

2. Lemmas

Lemma 2.1. *A ring map $A \to B$ such that B is generated by a finite number of elements integral over A is finite.*

Lemma 2.2. *A composition of finite ring maps is finite.*

Lemma 2.3. *If $A \to B$ is a finite ring map, then every element of B is integral over A.*

Lemma 2.4. *Let $A \to B$ be a ring map. The set of elements in B which are integral over A is an A-sub algebra of B*

Lemma 2.5. *Let $A \to B$ be an integral ring map and let $I \subset A$ be an ideal. Every element of IB is a root of a polynomial $P(T) = T^d + a_1 T^{d-1} + \ldots + a_d \in A[T]$ with $a_i \in I$.*

Lemma 2.6. *Let $A \subset B$ be an integral extension of domains and assume A is normal. Set $K = Frac(A)$ and $L = Frac(B)$. For any $b \in B$ the minimal polynomial of b over K has coefficients in A.*

Lemma 2.7. *Let A be a ring. Suppose $T^n + a_1 T^{n-1} + \ldots + a_n \in A[T]$ divides $T^m + b_1 T^{m-1} + \ldots + b_m \in A[T]$. Then $a_1, \ldots, a_n \in \sqrt{(b_1, \ldots, b_m)}$.*