The irreducible representations of $GL_2(\mathbb{F})$

June 2, 2014

Throughout, $\mathbb{F} = \mathbb{F}_q$ denotes the field with q elements, $q = p^n$ a prime power. Let $G = GL_2(\mathbb{F})$. To choose an element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of G, we must choose a nonzero first column $v = (a, c)$ for A, so there are $q^2 - 1$ choices for the first column. The second column can be any vector not a scalar multiple of v, and hence there are $q^2 - q$ choices for the second column once we have chosen the first. Thus,

$$\#(G) = (q^2 - 1)(q^2 - q) = q(q+1)(q-1)^2.$$

There are other groups associated to $GL_2(\mathbb{F})$. First, there is $SL_2(\mathbb{F})$, which by definition is the kernel of $\det\colon GL_2(\mathbb{F}) \to \mathbb{F}^*$. Since det is surjective, $\mathbb{F}^* \cong GL_2(\mathbb{F})/SL_2(\mathbb{F})$, from which it follows that

$$\#(\mathbb{F}^*) = \#(GL_2(\mathbb{F}))/\#(SL_2(\mathbb{F})).$$

Thus

$$\#(SL_2(\mathbb{F})) = \#(GL_2(\mathbb{F}))/(q-1) = q(q+1)(q-1).$$

There is also $PGL_2(\mathbb{F})$, which is the quotient of $GL_2(\mathbb{F})$ by its center $Z = \{aI : a \in \mathbb{F}^*\}$ and $PSL_2(\mathbb{F})$, which is the quotient of $SL_2(\mathbb{F})$ by *its* center $\{\pm I\}$. If $\operatorname{char} \mathbb{F} \neq 2$, then $\#(\{\pm I\}) = 2$, but if $\operatorname{char} \mathbb{F} = 2$, then $I = -I$ and $PSL_2(\mathbb{F}) = SL_2(\mathbb{F})$. From this, we see that

$$\#(PGL_2(\mathbb{F})) = q(q+1)(q-1)$$
$$\#(PSL_2(\mathbb{F})) = \begin{cases} \frac{1}{2}q(q+1)(q-1), & \text{if } q \text{ is odd;} \\ q(q+1)(q-1), & \text{if } q = 2^n. \end{cases}$$

The representation theory of all of these groups is closely related, and for simplicity we will just look at $GL_2(\mathbb{F})$. One reason for looking at $PSL_2(\mathbb{F})$, though, is the following theorem (which can be proved directly or using representation theory):

Theorem: If $q > 3$, then $PSL_2(\mathbb{F})$ is simple.

The orders of $PSL_2(\mathbb{F})$ for the first few values of q are given by the following table:

q	2	3	4	5	7
$\#(PSL_2(\mathbb{F}_q))$	6	12	60	60	168

As one might expect, $PSL_2(\mathbb{F}_2) \cong S_3$ and $PSL_2(\mathbb{F}_3) \cong A_4$, neither of which is simple. Moreover, $PSL_2(\mathbb{F}_4) \cong PSL_2(\mathbb{F}_5) \cong A_5$, which is simple. The group $PSL_2(\mathbb{F}_7)$ is a simple group which is not isomorphic to A_n for any n. In general, the simple groups of the form $PSL_2(\mathbb{F}_q)$ are the first case of *simple groups of Lie type*, a basic class of finite simple groups.

1 Certain subgroups of G

The following are important subgroups associated to G:

1. The *diagonal subgroup* $D = \left\{\begin{pmatrix} r & 0 \\ 0 & s\end{pmatrix} : r, s \in \mathbb{F}^*\right\}$ (also known as a *split Cartan subgroup*). It is isomorphic to the product $\mathbb{F}^* \times \mathbb{F}^*$. Hence $\#(D) = (q-1)^2$. Note that D contains the center
$$Z = Z(G) = \{aI : a \in \mathbb{F}^*\}$$
of G

2. The *Borel subgroup* $B = \left\{\begin{pmatrix} r & s \\ 0 & u\end{pmatrix} : r, u \in \mathbb{F}^*, s \in \mathbb{F}\right\}$. More invariantly,
$$B = \{A \in G : Ae_1 = ae_1 \text{ for some } a \in \mathbb{F}^*\}.$$
The subgroup $U = \left\{\begin{pmatrix} 1 & s \\ 0 & 1\end{pmatrix} : s \in \mathbb{F}\right\}$ is a normal subgroup isomorphic to $\mathbb{F}$. In fact, $\varphi_B \colon B \to \mathbb{F}^* \times \mathbb{F}^*$ defined by
$$\varphi_B\left(\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}\right) = (a, d)$$
is a surjective homomorphism from B to $\mathbb{F}^* \times \mathbb{F}^*$ and $\operatorname{Ker}\varphi_B = U$. Hence the quotient B/U is isomorphic to $\mathbb{F}^* \times \mathbb{F}^*$. Note that D is a

(non-normal) subgroup of B and in fact B is the semi-direct product of U and D. Finally,

$$\#(B) = q(q-1)^2 = \#(U)\#(D).$$

3. For simplicity we assume that $\operatorname{char}\mathbb{F} \neq 2$. We fix once and for all an element $\alpha \in \mathbb{F}$ such that $\alpha \notin (\mathbb{F})^2$. Then $\mathbb{F}(\sqrt{\alpha}) = \mathbb{F}'$ is the unique degree q extension of $\mathbb{F}$, and $\#(\mathbb{F}') = q^2$. It is straightforward to check that $H = \left\{ \begin{pmatrix} r & s\alpha \\ s & r \end{pmatrix} : r, s \in \mathbb{F}, \text{not both } 0 \right\}$ is a subgroup of G (also known as a *non-split Cartan subgroup*). The group H is isomorphic to the multiplicative group $(\mathbb{F}')^*$, via $\varphi_H \colon H \to (\mathbb{F}')^*$, where

$$\varphi_H\left(\begin{pmatrix} r & s\alpha \\ s & r \end{pmatrix}\right) = r + s\sqrt{\alpha}.$$

Thus $\#(H) = q^2 - 1$.

Geometrically, the Borel subgroup B arises as follows: Let $\mathbb{P}^1(\mathbb{F})$ be the projective line over $\mathbb{F}$. By definition, $\mathbb{P}^1(\mathbb{F})$ is the set of lines in $\mathbb{F}^2$. Thus an element of $\mathbb{P}^1(\mathbb{F})$ is an equivalence class $[v]$, where $v \in \mathbb{F}^2 - \{(0,0)\}$ and and two nonzero elements v_1 and v_2 are equivalent if there exists a $t \in \mathbb{F}^*$ such that $v_2 = tv_1$. Note that, if $a \neq 0$, then (a,b) is equivalent to $(1, b/a)$, and hence

$$\mathbb{P}^1(\mathbb{F}) = \{(1,t) : t \in \mathbb{F}\} \cup \{(0,1)\}.$$

In particular, $\#(\mathbb{P}^1(\mathbb{F})) = q+1$. Clearly G acts on $\mathbb{P}^1(\mathbb{F})$ via $A \cdot [v] = [Av]$, and the isotropy subgroup of $[v]$ is the subgroup

$$\{A \in G : Av = tv \text{ for some } t \in \mathbb{F}^*\}.$$

By definition, if $v = e_1 = (1,0)$, then $Ae_1 = te_1$ for some $t \in \mathbb{F}^* \iff A \in B$. Also, an element A of G acts trivially on $\mathbb{P}^1(\mathbb{F})$, i.e. $A[v] = [v]$ for all $[v] \in \mathbb{P}^1(\mathbb{F})$, $\iff$ A is a multiple of the identity, i.e. $A \in Z(G)$.

Since G acts transitively on $\mathbb{P}^1(\mathbb{F})$ and the isotropy subgroup of $[e_1]$ is B, there is a G-isomorphism of G-sets from G/B to $\mathbb{P}^1(\mathbb{F})$. In particular, $\#(G/B) = q+1$, which we can also see directly from

$$\#(G/B) = \#(G)/\#(B) = \frac{q(q-1)^2(q+1)}{q(q-1)^2} = q+1.$$

Finally, we note that G acts *doubly transitively* on $\mathbb{P}^1(\mathbb{F})$. In fact, if $[v_1]$ and $[v_2]$ are two elements of $\mathbb{P}^1(\mathbb{F})$ with $[v_1] \neq [v_2]$, then the vectors v_1 and

v_2 are linearly independent, so there is a (unique) $A \in G$ such that $Ae_1 = v_1$ and $Ae_2 = v_2$. It follows that G acts transitively on the set

$$\{([v_1], [v_2]) \in \mathbb{P}^1(\mathbb{F}) : [v_1] \neq [v_2]\},$$

and hence doubly transitively on $\mathbb{P}^1(\mathbb{F})$. Note that, in this context, the diagonal subgroup D is just the isotropy subgroup of the pair $([e_1], [e_2])$.

2 Conjugacy classes in $GL_2(\mathbb{F})$

In what follows, we divide the conjugacy classes of G into four possible types. For each type, we describe the elements A of that type, the order of the centralizer $Z_G(A)$, and hence the number of elements in the conjugacy class $C(A)$ (since there is a bijection from $G/Z_G(A)$ to $C(A)$, so that $\#(G/Z_G(A)) = \#(C(A))$). Finally, we list the number of conjugacy classes of the given type.

Type I: A is in the center $Z = Z(G)$ of G. In this case $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. By definition, $Z_G(A) = G$ and hence $\#(Z_G(A)) = \#(G)$, $\#(G/Z_G(A)) = \#(C(A)) = 1$. Finally, there are $q-1$ elements in $Z(G)$, since there are $q-1$ choices for $a \in \mathbb{F}^*$.

Type II: A has two distinct eigenvalues, in other words the characteristic polynomial $p_A(t)$ has two distinct roots in $\mathbb{F}$. (Note that Type I corresponds to the case where $p_A(t)$ has a repeated root in $\mathbb{F}$ and A is diagonalizable.) In this case, up to conjugation, $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $a \neq d$. It is straightforward to check that $X = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ commutes with A (i.e. $XA = AX$) $\iff$ $X = \begin{pmatrix} r & 0 \\ 0 & u \end{pmatrix}$ is also a diagonal matrix; this follows either by direct computation or by noting that X has to send the eigenvalue e_i to a scalar multiple of e_i, $i = 1, 2$. Hence $Z_G(A) = D$ and $\#(Z_G(A)) = (q-1)^2$, so that

$$\#(G/Z_G(A)) = \#(C(A)) = q(q+1)(q-1)^2/(q-1)^2 = q(q+1) = q^2 + q.$$

Note that A is conjugate to $A = \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$, since we cannot *a priori* order the possible eigenvalues (or eigenvectors); more directly,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}.$$

Thus the conjugacy class of A is specified by the unordered pair $\{a, d\}$, with $a, d \in \mathbb{F}^*$, $a \neq d$. There are $\binom{q-1}{2} = \frac{(q-1)(q-2)}{2}$ such unordered pairs, hence $\frac{(q-1)(q-2)}{2}$ such conjugacy classes. The total number of elements of Type II is

$$\frac{(q-1)(q-2)}{2} \cdot (q(q+1)) = \frac{1}{2}q(q+1)(q-1)(q-2).$$

Type III: The characteristic polynomial $p_A(t)$ has a repeated root in $\mathbb{F}$ and A is not diagonalizable. If $p_A(t) = (t-a)^2$, then $\operatorname{Ker} A - a \operatorname{Id}$ is one-dimensional and it is easy to see that we can choose a basis e_1, e_2 such that, in this basis, $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$. A computation shows that $X = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ commutes with A (i.e. $XA = AX$) $\iff$ $X = \begin{pmatrix} r & s \\ 0 & r \end{pmatrix}$. Hence $Z_G(A) \subseteq B$ and $\#(Z_G(A)) = q(q-1)$, so that

$$\#(G/Z_G(A)) = \#(C(A)) = q(q+1)(q-1)^2/q(q-1) = (q-1)(q+1) = q^2-1.$$

The conjugacy class of A is specified by the repeated root $a \in \mathbb{F}^*$ of $p_A(t)$, hence there are $q-1$ such conjugacy classes. The total number of elements of Type III is then

$$(q-1)(q-1)(q+1) = (q-1)^2(q+1).$$

Type IV: There are no roots of the characteristic polynomial $p_A(t)$ in $\mathbb{F}$. Under the simplifying assumption that $\operatorname{char} \mathbb{F} \neq 2$, we choose α as in Section 1. Let $\gamma = a + b\sqrt{\alpha}$ be a root of $p_A(t)$, with $b \neq 0$. To find an example of an A such that γ is a root of $p_A(t)$, one can look at the matrix representation of multiplication of γ on $\mathbb{F}(\sqrt{\alpha})$, which is a two-dimensional $\mathbb{F}$-vector space, using the basis $1, \sqrt{\alpha}$. In general, one can check directly that A is conjugate to the matrix $A = \begin{pmatrix} a & b\alpha \\ b & a \end{pmatrix}$, where $b \neq 0$. This also follows by counting the total number of elements of G conjugate to a matrix of the form A above and comparing this with the order of G.. Next, a calculation shows that $X = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ commutes with A (i.e. $XA = AX$) $\iff$ $X = \begin{pmatrix} r & s\alpha \\ s & r \end{pmatrix}$. The condition that $X \in G = GL_2(\mathbb{F})$, i.e. that $\det X \neq 0$, is just the condition that not both of r and s are 0. Hence $Z_G(A) = H$ and $\#(Z_G(A)) = q^2 - 1$, so that

$$\#(G/Z_G(A)) = \#(C(A)) = q(q+1)(q-1)^2/q^2 - 1 = q(q-1) = q^2 - q.$$

The total number of conjugacy classes is the number of possible $\gamma = a + b\sqrt{\alpha}$ which are roots of $p_A(t)$. However, if γ is a root of $p_A(t)$, then so is $\bar{\gamma} = a - b\sqrt{\alpha}$. So the conjugacy classes are indexed by the conjugate pairs $\{\gamma, \bar{\gamma}\}$, where $\gamma \in \mathbb{F}'$ but $\gamma \notin \mathbb{F}$. In fact, one can also see directly that $A = \begin{pmatrix} a & b\alpha \\ b & a \end{pmatrix}$ and $\bar{A} = \begin{pmatrix} a & -b\alpha \\ -b & a \end{pmatrix}$ are conjugate, by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The total number of conjugacy classes is then $\frac{1}{2}(q^2 - q)$, and the total number of elements of Type IV is then

$$\frac{1}{2}(q^2 - q)(q^2 - q) = \frac{1}{2}q^2(q-1)^2.$$

As a check on our calculations, if we add up the number of elements of Types I, II, III, IV, we get

$$\begin{aligned}
(q-1) + \frac{1}{2}q(q+1)(q-1)(q-2) + (q-1)^2(q+1) + \frac{1}{2}q^2(q-1)^2 \\
= \frac{1}{2}(q-1)\left[2 + q(q+1)(q-2) + 2(q-1)(q+1) + q^2(q-1)\right] \\
= \frac{1}{2}(q-1)\left[2 + q^3 - q^2 - 2q + 2q^2 - 2 + q^3 - q^2\right] \\
= \frac{1}{2}(q-1)(2q^3 - 2q) = (q-1)q(q+1)(q-1) = \#(G).
\end{aligned}$$

Finally, we tally the number h of conjugacy classes of G:

$$\begin{aligned}
h = (q-1) + \frac{(q-1)(q-2)}{2} + (q-1) + \frac{1}{2}(q^2 - q) \\
= (q-1)\left[1 + \frac{(q-2)}{2} + 1 + \frac{1}{2}q\right] = (q-1)(q+1) = q^2 - 1.
\end{aligned}$$

3 Construction of irreducible representations

We need to construct $q^2 - 1$ pairwise non-isomorphic irreducible representations.

One dimensional representations: We have the determinant homomorphism $\det\colon G \to \mathbb{F}^*$. If $f\colon \mathbb{F}^* \to \mathbb{C}^*$ is a homomorphism, then $f \circ \det\colon G \to$

$\mathbb{C}^*$ is a homomorphism, and hence corresponds to a one dimensional representation V_f with character $\chi_{V_f} = f \circ \det$. Since $\mathbb{F}$ is a finite field, $\mathbb{F}^*$ is a cyclic group of order $q-1$. Hence there are $q-1$ possible homomorphisms f and $q-1$ irreducible one dimensional representations obtained in this way. In fact, every one dimensional representation is of this form. This will follow from our list of representations, or can be checked by showing that the commutator subgroup of G is $SL_2(\mathbb{F})$.

For completeness, we record the character table of such representations:

A	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $a \neq d$	$A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & b\alpha \\ b & a \end{pmatrix}$, $b \neq 0$
$\chi_{V_f}(A)$	$f(a)^2$	$f(a)f(d)$	$f(a)^2$	$f(a^2 - b^2\alpha)$

A permutation representation and related representations: The group G acts on $\mathbb{P}^1(\mathbb{F})$, so we can form the associated permutation representation $\mathbb{C}[\mathbb{P}^1(\mathbb{F})]$. Thus we can write

$$\mathbb{C}[\mathbb{P}^1(\mathbb{F})] \cong \mathbb{C} \oplus W,$$

where $\mathbb{C}$ is the trivial subrepresentation of $\mathbb{C}[\mathbb{P}^1(\mathbb{F})]$, with basis equal to the vector $\sum_{[v]\in\mathbb{P}^1(\mathbb{F})}[v]$, and W is a complement. For example we could take W to be the G-invariant subspace

$$W = \left\{ \sum_{[v]\in\mathbb{P}^1(\mathbb{F})} a_{[v]}[v] : \sum_{[v]\in\mathbb{P}^1(\mathbb{F})} a_{[v]} = 0 \right\}.$$

By what we have seen W is irreducible since the G-action is doubly transitive. We can also compute the character of W explicitly: let χ be the character of the representation $\mathbb{C}[\mathbb{P}^1(\mathbb{F})]$. Then $\chi(A)$ is the number of fixed points of A. But a fixed point of A is the same as the line spanned by a nonzero eigenvector of A. It follows that

1. $\chi(A) = q + 1$ if $A \in Z(G)$.
2. $\chi(A) = 2$ if $A \in D$, $A \notin Z(G)$.
3. $\chi(A) = 1$ if $A \in B$, $A \notin D$
4. $\chi(A) = 0$ if $A \in H$.

Since $\chi_W = \chi - 1$, we have the following table for the character χ_W of W:

Type of A	I	II	III	IV
$\chi_W(A)$	q	1	0	-1

As a check, one can work out directly that $\langle \chi_W, \chi_W \rangle = 1$, so that W is irreducible.

Note that, if $\varepsilon = f \circ \det$ is a homomorphism from G to $\mathbb{C}^*$ as above, then we can form the associated representation $W \otimes \varepsilon = W_f$, with character $\chi_{W_f} = \varepsilon \chi_W$, and these are all distinct since, for example, taking $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ with $a \neq 1$, $\chi_{W_f}(A) = f(a)$, and hence we can recover f and therefore ε from the character. Thus, in all, there are $q-1$ irreducible representations of dimension q obtained in this way, with $W = W_1$. The character tables are as follows:

A	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $a \neq d$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & b\alpha \\ b & a \end{pmatrix}$, $b \neq 0$
$\chi_{W_f}(A)$	$qf(a)^2$	$f(a)f(d)$	0	$-f(a^2 - b^2\alpha)$

Induced representations from the Borel subgroup: Recall that we have a surjective homomorphism $\varphi_B \colon B \to \mathbb{F}^* \times \mathbb{F}^*$. Hence, given two homomorphisms $f_1 \colon \mathbb{F}^* \to \mathbb{C}^*$ and $f_2 \colon \mathbb{F}^* \to \mathbb{C}^*$, there is a homomorphism $(f_1, f_2) \colon \mathbb{F}^* \times \mathbb{F}^* \to \mathbb{C}^*$ defined by

$$(f_1, f_2)(a, d) = f_1(a) f_2(d).$$

Composing with φ_B, we get a homomorphism $(f_1, f_2) \circ \varphi_B$ and thus a corresponding one dimensional representation L_{f_1,f_2} of B.

Let $U_{f_1,f_2} = \operatorname{Ind}_B^G L_{f_1,f_2}$. We claim:

Theorem: The representation U_{f_1,f_2} is irreducible $\iff$ $f_1 \neq f_2$. Moreover, $U_{f_1,f_2} \cong U_{f_1',f_2'} \iff f_1 = f_1'$ and $f_2 = f_2'$ or $f_1 = f_2'$ and $f_2 = f_1'$.

As we shall see, since there are $q-1$ homomorphisms $f \colon \mathbb{F}^* \to \mathbb{C}^*$, this gives $\frac{1}{2}(q-1)(q-2)$ new representations, all of dimension

$$\#(G)/\#(B) \dim L_{f_1,f_2} = q + 1.$$

First, we deal with the case $f_1 = f_2 = f$, say. In this case,

$$(f_1, f_2)(a, d) = f_1(a) f_2(d) = f(a) f(d) = f(ad)$$

and hence $(f_1, f_2) \circ \varphi_B = f \circ \det$ on B. In other words, $L_{f,f} = \operatorname{Res}_B^G V_f$, where V_f is defined above. Thus

$$U_{f,f} = \operatorname{Ind}_B^G \operatorname{Res}_B^G V_f \cong V_f \otimes \mathbb{C}[G/B] = V_f \otimes \mathbb{C}[\mathbb{P}^1(\mathbb{F})].$$

Since $\mathbb{C}[\mathbb{P}^1(\mathbb{F})] \cong \mathbb{C} \oplus W$, we see that

$$U_{f,f} \cong V_f \oplus W_f.$$

Next, we see what Mackey's theorem says about the irreducibility of U_{f_1,f_2}. In fact, by Mackey's theorem, U_{f_1,f_2} is irreducible $\iff$ for all $A \notin B$, if we set $B_A = ABA^{-1} \cap B$, then the two representations $\operatorname{Res}_{B_A}^B L_{f_1,f_2}$ and $\operatorname{Res}_{B_A}^{ABA^{-1}} L_{f_1,f_2}^A$ are disjoint (have no irreducible factors in common), where L_{f_1,f_2}^A is the one-dimensional representation on ABA^{-1} corresponding to the character $(f_1, f_2) \circ \varphi_B \circ i_{A^{-1}}$. We shall use the following:

Lemma: Let $A \in G$.

1. $ABA^{-1} \cap B = B$ if $A \in B$.
2. $ABA^{-1} \cap B$ is conjugate in B to D if Ae_1 is not a multiple of e_1.
3. If $A \notin B$, then A is in the double coset $B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B$, in other words $A = A_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_2$ for some $A_1, A_2 \in B$.

Proof: Clearly, if $A \in B$ then $ABA^{-1} = B$ and hence $ABA^{-1} \cap B = B$. Note that $A \in B \iff Ae_1$ is not a multiple of e_1. In this case, let $v = Ae_1$. Then the two lines $[e_1]$ and $[v]$ are different points of $\mathbb{P}^1(\mathbb{F})$, and by definition $ABA^{-1} \cap B$ is the stabilizer of the ordered pair $([e_1], [v])$. Then $ABA^{-1} \cap B$ is conjugate to D by any element of B which takes e_1 to a multiple of e_1 and $[v]$ to $[e_2]$. Finally, if $X_1 \in B$ is such that $X_1 v = e_2$, then $X_1 A e_1 = e_2$, hence $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X_1 A e_1 = e_1$. It follows that

$$A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X_1 A \in B,$$

in other words that

$$A = X_1^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_2 = A_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_2,$$

where A_1 and $A_2 \in B$. $\square$

Using the lemma, we check that the hypotheses of Mackey's theorem are fulfilled. First we consider the case where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A^{-1}$. Then $B_A = D$, and $\mathrm{Res}^B_{B_A} L_{f_1,f_2}$ just corresponds to the homomorphism $D \to \mathbb{C}^*$ which maps $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ to $f_1(a)f_2(d)$. On D, $(f_1, f_2) \circ \varphi_B \circ i_{A^{-1}} = (f_2, f_1) \circ \varphi_B$ since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}.$$

More generally, if $A = A_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_2$, where A_1 and $A_2 \in B$, then

$$(f_1, f_2) \circ \varphi_B \circ i_{A^{-1}} = (f_1, f_2) \circ \varphi_B \circ i_{A_2^{-1}} \circ i_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} \circ i_{A_1^{-1}}$$
$$= (f_1, f_2) \circ \varphi_B \circ i_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} \circ i_{A_2^{-1}} = (f_2, f_1) \circ \varphi_B \circ i_{A_1^{-1}} = (f_2, f_1) \circ \varphi_B,$$

where we have used the fact that $\varphi_B \circ i_X = \varphi_B$ if $X \in B$, since the image of φ_B is abelian. We see then that the characters for the one dimensional representations $\mathrm{Res}^B_{B_A} L_{f_1,f_2}$ and $\mathrm{Res}^{ABA^{-1}}_{B_A} L^A_{f_1,f_2}$ are disjoint $\iff$ the homomorphisms (f_1, f_2) and (f_2, f_1) are different $\iff$ $f_1 \neq f_2$.

Let us compute the character of U_{f_1,f_2}. Recall that

$$\chi_{\mathrm{Ind}^G_B L_{f_1,f_2}}(A) = \frac{1}{\#(B)} \sum_{X^{-1}AX \in B} \chi_{L_{f_1,f_2}}(X^{-1}AX).$$

We tabulate the possibilities:

Lemma:

1. If $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in Z(G)$, then $X^{-1}AX = A \in B$ for all $X \in G$.
2. If $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $a \neq d$, then $X^{-1}AX \in B \iff$ either $X \in B$ or $X \in \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B$.
3. If $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, then $X^{-1}AX \in B \iff X \in B$.
4. If the eigenvalues of A are not in $\mathbb{F}$, i.e. A is of Type IV, then for all $X \in G$, $X^{-1}AX$ is not in B.

Proof: (1) is clear. For the remaining cases, note that $X^{-1}AX \in B \iff X^{-1}AXe_1 = ae_1$ for some $a \in \mathbb{F}^*$, $\iff A(Xe_1) = aXe_1$ for some $a \in \mathbb{F}^*$, $\iff Xe_1$ is an eigenvector of A. If $A \in D$, this says either that Xe_1 is a multiple of e_1, and hence that $X \in B$, or that Xe_1 is a multiple of e_2. Since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ switches e_1 and e_2, this says that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X \in B$, hence $X \in \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B$. This proves (2), and (3) is similar but simpler. Finally, in case (4), the eigenvalues of $X^{-1}AX$ are the same as those for A and hence do not lie in $\mathbb{F}$. Thus $X^{-1}AX$ is never in B. □

This gives the following list for the values of $\chi_{U_{f_1,f_2}}$:

A	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, a \neq d$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & b\alpha \\ b & a \end{pmatrix}, b \neq 0$
$\chi_{U_{f_1,f_2}}$	$(q+1)f_1(a)f_2(a)$	$f_1(a)f_2(d) + f_1(d)f_2(a)$	$f_1(a)f_2(a)$	0

Since the values above are symmetric with respect to f_1 and f_2, we see that $\chi_{U_{f_1,f_2}} = \chi_{U_{f_2,f_1}}$ and hence that $U_{f_1,f_2} \cong U_{f_2,f_1}$. Also, by considering the values of $\chi_{U_{f_1,f_2}}$ on $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and on $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, we see that the function $\chi_{U_{f_1,f_2}}$ determines the product $f_1(a)f_2(a)$ and the sum $f_1(a) + f_2(a)$, and hence determines f_1, f_2 up to permutation.

Note also that once we have a formula for the character $\chi_{U_{f_1,f_2}}$, we could check directly that U_{f_1,f_2} is irreducible for $f_1 \neq f_2$ by showing that $\langle \chi_{U_{f_1,f_2}}, \chi_{U_{f_1,f_2}} \rangle = 1$. Likewise, in case $f_1 = f_2 = f$, we could verify directly from the character tables that $\chi_{U_{f,f}} = \chi_{V_f} + \chi_{W_f}$.

Thus, in all, we obtain $\binom{q-1}{2} = \frac{1}{2}(q-1)(q-2)$ irreducible representations in this way, all of dimension $q+1$.

The remaining representations: These are harder to describe explicitly. We begin by inducing a one dimensional representation of H to G. Let $\theta\colon (\mathbb{F}')^* \to \mathbb{C}^*$ be a homomorphism, where as before $\mathbb{F}' = \mathbb{F}(\alpha)$. Then θ corresponds to a one dimensional representation N_θ of H, since $H \cong (\mathbb{F}')^*$. Then $\operatorname{Ind}_H^G N_\theta$ is a representation of G, which however is **not** irreducible. For brevity, we denote by χ_θ the character $\chi_{\operatorname{Ind}_H^G N_\theta}$.

To deal with this problem, we consider the subgroup ZU consisting of all products of a scalar matrix aI with an element of U, where Z is the center of G; thus

$$ZU = \left\{ \begin{pmatrix} a & ab \\ 0 & a \end{pmatrix} : a \in \mathbb{F}^*, b \in \mathbb{F} \right\}.$$

It is easy to check that the map $(a, b) \mapsto \begin{pmatrix} a & ab \\ 0 & a \end{pmatrix}$ defines an isomorphism from $\mathbb{F}^* \times \mathbb{F}$ to ZU. In particular, if $f \colon \mathbb{F}^* \to \mathbb{C}^*$ and $g \colon \mathbb{F} \to \mathbb{C}^*$ are homomorphisms, then there is an induced one dimensional representation of $Z(G)U$, which we denote by $M_{f,g}$, and hence a representation $\operatorname{Ind}_{ZU}^G M_{f,g}$ of G, of dimension $\#(G)/\#(ZU) = (q-1)(q+1) = q^2-1$. We also denote by $\chi_{f,g}$ the character $\chi_{\operatorname{Ind}_{ZU}^G M_{f,g}}$. We are interested in the case where $f = \theta|\mathbb{F}^*$ is the restriction of θ to the subgroup $\mathbb{F}^*$ of $(\mathbb{F}')^*$. Also, we let $\sigma \colon \mathbb{F}' \to \mathbb{F}'$ be "conjugation:"

$$\sigma(a + b\sqrt{\alpha}) = a - b\sqrt{\alpha}.$$

Since σ is the nontrivial element of $\operatorname{Gal}(\mathbb{F}'/\mathbb{F})$, σ is the Frobenius homomorphism: $\sigma(\beta) = \beta^q$ for all $\beta \in \mathbb{F}'$.

Now suppose that $g \neq 1$. Then calculations for induced representations give the following values for $\chi_{\theta|\mathbb{F}^*,g}$ and χ_θ:

A	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $a \neq d$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & b\alpha \\ b & a \end{pmatrix}$, $b \neq 0$
$\chi_{\theta\|\mathbb{F}^*,g}$	$(q^2-1)\theta(a)$	0	$-\theta(a)$	0
χ_θ	$(q^2-q)\theta(a)$	0	0	$\theta(\varphi_H(A)) + \theta \circ \sigma(\varphi_H(A))$

where $\varphi_H \colon H \to (\mathbb{F}')^*$ is the homomorphism sending $\begin{pmatrix} a & b\alpha \\ b & a \end{pmatrix}$ to the element $a + b\sqrt{\alpha}$. (In particular, the character $\chi_{\theta|\mathbb{F}^*,g}$, and more generally the character $\chi_{f,g}$, do not depend on g, so that all of the induced representations $\operatorname{Ind}_{ZU}^G M_{f,g}$ are isomorphic, provided that $g \neq 1$.)

Now consider the class function ψ_θ defined by

$$\psi_\theta = \chi_{\theta|\mathbb{F}^*,g} - \chi_\theta.$$

It is clearly a combination of characters of irreducible representations with integer coefficients, not necessarily positive: $\psi_\theta = \sum_{i=1}^h n_i \chi_i$, where $h = q^2 - 1$ is the number of conjugacy classes, $n_i \in \mathbb{Z}$, and the χ_i are the characters of the irreducible representations of G. By looking at the table above, we see that the values of ψ_θ are given as follows:

A	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $a \neq d$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & b\alpha \\ b & a \end{pmatrix}$, $b \neq 0$
ψ_θ	$(q-1)\theta(a)$	0	$-\theta(a)$	$-\theta(\varphi_H(A)) - \theta\circ\sigma(\varphi_H(A))$

A computation shows that $\langle \psi_\theta, \psi_\theta \rangle = 1$, provided that $\theta \neq \theta \circ \sigma$. Thus, in the expression $\psi_\theta = \sum_{i=1}^h n_i \chi_i$, since $\langle \psi_\theta, \psi_\theta \rangle = \sum_{i=1}^h n_i^2$, there exists exactly one i such that $n_i \neq 0$, and in fact $n_i = \pm 1$. But since $\chi_i(1) = d_i > 0$ and $\psi_\theta(1) = q - 1 > 0$, we see that ψ_θ is the character of an irreducible representation O_θ.

For a character θ, if β is a generator of the cyclic group $(\mathbb{F}')^*$, then $\theta(\beta) = \zeta$ for some $(q^2-1)^{\text{st}}$ root of unity ζ, and hence $\theta(\sigma(\beta)) = \theta(\beta^q) = \zeta^q$. Then $\theta \neq \theta \circ \sigma \iff \zeta^q \neq \zeta$. Since $\zeta^q = \zeta \iff \zeta$ is a $(q-1)^{\text{st}}$ root of unity, there are $(q^2 - 1) - (q - 1) = q^2 - q$ choices for ζ, and hence for θ. Clearly $\psi_\theta = \psi_{\theta\circ\sigma}$, and hence $O_\theta \cong O_{\theta\circ\sigma}$. Moreover, O_θ determines ψ_θ and hence the values $\theta(a)$, $a \in \mathbb{F}^*$ as well as $\theta(\beta) + \theta(\sigma(\beta))$ for all $\beta \in (\mathbb{F}')^*$, for example for β a generator of the cyclic group $(\mathbb{F}')^*$. In particular, since $\theta(\beta\theta(\sigma(\beta)) = \theta(\beta\sigma(\beta)) = \theta(a)$ for $a = \beta\sigma(\beta) \in \mathbb{F}^*$. Thus the representation O_θ determines the unordered pair $\{\theta, \theta\circ\sigma\}$. So we see that the total number of different representations obtained in this way is $\frac{1}{2}(q^2 - q)$.

Counting up all four types of representations, we see that we have found

$$(q-1) + (q-1) + \frac{1}{2}(q-1)(q-2) + \frac{1}{2}(q^2 - q) = (q-1)(2 + \frac{q-2}{2} + \frac{q}{2}) \\ = (q-1)(q+1) = q^2 - 1.$$

Thus we have found all of the irreducible representations.