Representations

1 Basic definitions

If V is a k-vector space, we denote by $\operatorname{Aut} V$ the group of k-linear isomorphisms $F\colon V \to V$ and by $\operatorname{End} V$ the k-vector space of k-linear maps $F\colon V \to V$. Thus, if $V = k^n$, then $\operatorname{Aut} V \cong GL(n,k)$ and $\operatorname{End} V \cong \mathbb{M}_n(k)$. In general, if V is a finite dimensional vector space of dimension n, then a choice of basis defines a group isomorphism $\operatorname{Aut} V \cong GL(n,k)$ and a vector space $\operatorname{End} V \cong \mathbb{M}_n(k)$.

From now on, unless otherwise stated, $k = \mathbb{C}$, i.e all vector spaces are $\mathbb{C}$-vector spaces, all linear maps are $\mathbb{C}$-linear, and all vector subspaces are closed under scalar multiplication by $\mathbb{C}$.

Definition 1.1. Let G be a group. A *representation of G on V* is a homomorphism $\rho\colon G \to \operatorname{Aut} V$, where V is a finite dimensional $\mathbb{C}$-vector space. Equivalently, for all $g \in G$, $\rho(g)\colon V \to V$ is a linear map satisfying: $\rho(g)(\rho(h)(v)) = \rho(gh)(v)$ and $\rho(1) = \mathrm{Id}$, i.e. a representation is equivalent to an action of G on V by linear maps. The *degree* $\deg \rho$ is by definition $\dim V$. Finally, a choice of basis of V identifies ρ with a homomorphism (also denoted ρ) from G to $GL(n,\mathbb{C})$. Changing the basis replaces ρ by $T\rho T^{-1}$, where T is an invertible matrix.

We will usually abbreviate the data of the representation $\rho\colon G \to \operatorname{Aut} V$ by ρ, or frequently by V, with the understanding that the vector space V includes the data of the homomorphism ρ or of the G-action. Given V, we often denote the corresponding homomorphism by ρ_V, especially if there are several different G-representations under discussion.

Remark 1.2. For a general field k and a finite dimensional k-vector space V (and we are especially interested in the case $k = \mathbb{R}$ or $k = \mathbb{Q}$), we can speak of a *k-representation*, i.e. a homomorphism $G \to \operatorname{Aut} V$. After choosing a k-basis, this amounts to a homomorphism $G \to GL(n,k)$. Note that, if k is a subfield of a larger field K, there is an obvious inclusion $GL(n,k) \subseteq$

$GL(n,K)$ which realizes $GL(n,k)$ as a subgroup of $GL(n,K)$. For example, the group of $n \times n$ invertible matrices with real (or rational) coefficients is a subgroup of $GL(n,\mathbb{C})$. To say that V is a real representation, or a rational representation of G, is to say that we can find a representation and an appropriate basis so that all of the corresponding matrices have real or rational entries. As we will see, this is not always possible.

Conversely, the field $\mathbb{C}$ is also a vector space over $\mathbb{R}$ of dimension 2, with basis $1, i$. Similarly, $\mathbb{C}^n$ is a real vector space of dimension $2n$, with $\mathbb{R}$-basis $e_1, ie_1, e_2, ie_2, \dots, e_n, ie_n$. Moreover, a $\mathbb{C}$-linear map $F\colon \mathbb{C}^n \to \mathbb{C}^n$ is clearly $\mathbb{R}$-linear as well, giving an inclusion homomorphism $\iota\colon GL(n,\mathbb{C}) \to GL(2n,\mathbb{R})$. For example, in case $n = 1$, $GL(1,\mathbb{C}) = \mathbb{C}^*$, and the above homomorphism is given by

$$\iota(a+bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

For a general **finite extension** K of a field k (in case you have taken Modern Algebra II), we can view K as a k-vector space whose dimension over k is by definition the field degree $[K:k]$. Then K^n can be viewed as a k-vector space of dimension $[K:k]n$.

Example 1.3. (0) For $V = \{0\}$ the vector space of dimension zero, $\operatorname{Aut} V = \{\mathrm{Id}\}$ and there is a unique G-representation on V. However, this representation is uninteresting and we will systematically exclude it most of the time.

(1) $\dim V = 1$, $\rho(g) = \mathrm{Id}$ for all $g \in G$. We call V the *trivial representation*. More generally, we can take $\dim V$ arbitrary but still set $\rho(g) = \mathrm{Id}$ for all $g \in G$ (i.e. $\rho\colon G \to \operatorname{Aut} V$ is the trivial homomorphism). Unlike the case of the zero representation above, the trivial representation plays an important role.

(2) If $\dim V = 1$, $\operatorname{Aut} V \cong \mathbb{C}^*$ acting by multiplication, and a representation ρ is the same as a homomorphism $G \to \mathbb{C}^*$.

(3) If G is finite, then the *regular representation* ρ_{reg} is defined as follows: $V = \mathbb{C}[G]$, the free vector space with basis G, and the homomorphism ρ_{reg} is defined by:

$$\rho_{\mathrm{reg}}(h)\left(\sum_{g \in G} t_g \cdot g\right) = \sum_{g \in G} t_g \cdot (hg) = \sum_{g \in G} t_{h^{-1}g} \cdot g.$$

Viewing $\mathbb{C}[G]$ as the space of functions $f\colon G \to \mathbb{C}$, the G-action is given by

$$\rho(g)(f) = f \circ L_g^{-1},$$

where $L_g\colon G \to G$ is left multiplication by g. Notice that we take L_g^{-1} not L_g, which is necessary to keep the order right, and that this is already built into the above formula for ρ_{reg}.

(4) More generally, if X is a finite G-set, then $\mathbb{C}[X]$ is a G-representation via

$$\rho(h)\left(\sum_{x\in X} t_x \cdot x\right) = \sum_{x\in X} t_x \cdot (h\cdot x) = \sum_{x\in X} t_{h^{-1}x}\cdot x.$$

Again, we can view $\mathbb{C}[X]$ as the vector space of functions $f\colon X \to \mathbb{C}$, and the action on functions is given by $\rho(g)(f) = f \circ L_g^{-1}$, where $L_g\colon X \to X$ is the function defined by $L_g(x) = g\cdot x$.

For example, $\mathbb{C}^n$ is a representation of the symmetric group S_n (the *standard representation*) via: if $\sigma \in S_n$, then $\rho(\sigma)(e_i) = e_{\sigma(i)}$, and hence

$$\rho(\sigma)\left(\sum_{i=1}^n t_i e_i\right) = \sum_{i=1}^n t_i e_{\sigma(i)} = \sum_{i=1}^n t_{\sigma^{-1}(i)} e_i.$$

(5) If $G = \mathbb{Z}$, then a homomorphism $\rho\colon G \to GL(n,\mathbb{C})$ is uniquely determined by $\rho(1) = A$, since then $\rho(n) = A^n$ for all $n \in \mathbb{Z}$.

(6) If ρ is a G-representation and H is a subgroup of G, then we can restrict the function ρ to H to obtain a homomorphism $\rho|H\colon H \to \operatorname{Aut} V$. We denote this representation of H by $\operatorname{Res}_H^G \rho$. More generally, if $f\colon G' \to G$ is a homomorphism, then $\rho\circ f\colon G' \to \operatorname{Aut} V$ is a G'-representation.

2 Invariant subspaces and morphisms

Definition 2.1. A vector subspace of a G-representation V is a *G-invariant subspace* if, for all $g \in G$, $\rho(g)(W) = W \iff$ for all $g \in G$, $\rho(g)(W) \subseteq W$ (since then we also have $\rho(g^{-1})(W) = (\rho(g)^{-1})(W) \subseteq W$, hence $W \subseteq \rho(g)(W)$). In this case, W is also a G-representation via the action of G on V: For $w \in W$, we set $\rho_W(w) = \rho_V(w)$.

In the definition, we allow for the possibility that $W = \{0\}$, see (1) below.

Example 2.2. (1) $\{0\}$ and V are always G-invariant subspaces.

(2) For the standard representation $\mathbb{C}^n$ of S_n, $W_1 = \{(t,\dots,t) : t \in \mathbb{C}\}$ and $W_2 = \{(t_1,\dots,t_n) : \sum_{i=1}^n t_i = 0\}$ are S_n-invariant.

(3) If W is a one dimensional G-invariant subspace of V, then $W = \mathbb{C}\cdot v$ where v is a (nonzero) common eigenvector for G, i.e. $\rho(g)(v) = \lambda(g)v$ for

some $\lambda(g) \in \mathbb{C}^*$, and necessarily $\lambda\colon G \to \mathbb{C}^*$ is a homomorphism. Conversely, if v is a (nonzero) common eigenvector for G, then $\mathbb{C} \cdot v$ is a one dimensional G-invariant subspace of V.

(4) An easy argument shows that the intersection of two G-invariant subspaces is again G-invariant.

(5) If W_1 is a G-invariant subspace of V and W_2 is a G-invariant subspace of W_1, then clearly W_2 is a G-invariant subspace of V.

Definition 2.3. For a G-representation V, V^G is defined as for G-sets:

$$V^G = \{v \in V : \rho(g)(v) = v \text{ for all } g \in G\}.$$

It is a vector subspace of V, in fact a G-invariant subspace (possibly $\{0\}$). For example, $(\mathbb{C}^n)^{S_n} = W_1$ in the above notation.

Definition 2.4. If V_1 and V_2 are two G-representations, a *G-morphism* or simply a *morphism* or an *intertwining operator* is a linear map $F\colon V_1 \to V_2$ such that, for all $g \in G$, $F \circ \rho_{V_1}(g) = \rho_{V_2}(g) \circ F$. Equivalently, for all $g \in G$,

$$\rho_{V_2}(g) \circ F \circ \rho_{V_1}(g)^{-1} = F.$$

The composition of two G-morphisms is a G-morphism. The set of all G-morphisms $F\colon V \to W$ is clearly a vector subspace of $\mathrm{Hom}(V, W)$; we denote it by $\mathrm{Hom}^G(V, W)$. The function F is a *G-isomorphism* or simply an *isomorphism* if F is a linear isomorphism; in this case F^{-1} is also a G-morphism. The composition of two G-isomorphisms is a G-isomorphism. We use the symbol $\cong$ to denote G-isomorphism if the meaning is clear from the context.

Example 2.5. (1) If V is a G-representation and $t \in \mathbb{C}$, then $t\,\mathrm{Id}\colon V \to V$ is a G-isomorphism.

(2) If G is abelian and V is a G-representation, then $\rho_V(h)$ is a G-isomorphism from V to itself for all $h \in G$, because, for all $g \in G$,

$$\rho_V(h) \circ \rho_V(g) = \rho_V(hg) = \rho_V(gh) = \rho_V(g) \circ \rho_V(h).$$

More generally, for an arbitrary G, if V is a G-representation, then $\rho_V(h)$ is a G-isomorphism for all $h \in Z(G)$.

(3) For $G = S_n$, $V_1 = \mathbb{C}^n$ with the usual permutation representation of S_n, and $V_2 = \mathbb{C}$ viewed as the trivial representation of S_n, the linear map $F(t_1, \dots, t_n) = \sum_{i=1}^n t_i$ is an S_n-morphism of representations.

We leave the following as an exercise:

Lemma 2.6. *If V_1 and V_2 are two G-representations and $F\colon V_1 \to V_2$ is a G-morphism, then* $\operatorname{Ker} F$ *is a G-invariant subspace of V_1 and* $\operatorname{Im} F$ *is a G-invariant subspace of V_2. More generally, for every G-invariant subspace W_2 of V_2, $F^{-1}(W_2)$ is a G-invariant subspace of V_1, and, for every G-invariant subspace W_1 of V_1, $F(W_1)$ is a G-invariant subspace of V_2.* □

3 New G-representations from old

In this section, we describe how the standard linear algebra constructions lead to methods of constructing representations.

(1) If V_1 and V_2 are G-representations, then $V_1 \oplus V_2$ is also a representation, via:

$$\rho_{V_1 \oplus V_2}(g)(v_1, v_2) = (\rho_{V_1}(g)(v_1), \rho_{V_2}(g)(v_2)).$$

In terms of matrices, for appropriate choices of bases, $\rho_{V_1 \oplus V_2}(g)$ is written in block diagonal form:

$$\rho_{V_1 \oplus V_2}(g) = \begin{pmatrix} \rho_{V_1}(g) & O \\ O & \rho_{V_2}(g) \end{pmatrix}.$$

We are especially interested in internal direct sums. In fact, we have the following:

Lemma 3.1. *Let V be a G-representation. If W_1, W_2 are two G-invariant subspaces of V such that V is the (internal) direct sum of the subspaces W_1 and W_2, then the direct sum map $W_1 \oplus W_2 \to V$ is a G-isomorphism.*

Proof. The linear map $F\colon W_1 \oplus W_2 \to V$ defined by $F(w_, w_2) = w_1 + w_2$ is a linear isomorphism, and we have to show that it is a G-morphism. Note that, by definition, if $w_i \in W_i$, then $\rho_V(w_i) = \rho_{W_i}(w_i)$. Then

$$\begin{aligned}\rho_V \circ F(w_1, w_2) = \rho_V(w_1 + w_2) &= \rho_V(w_1) + \rho_V(w_2) = \rho_{W_1}(w_1) + \rho_{W_2}(w_2) \\ &= F(\rho_{W_1}(w_1), \rho_{W_2}(w_2)) = F \circ \rho_{W_1 \oplus W_2}(w_1, w_2).\end{aligned}$$

Thus F is a G-morphism. □

(2) If V is a G-representation, then $V^* = \operatorname{Hom}(V, \mathbb{C})$ is also a G-representation via

$$\rho_{V^*}(g)(f) = f \circ \rho_V(g^{-1}) = f \circ (\rho_V(g)^{-1}).$$

The inverse is necessary to keep ρ_{V^*} a homomorphism.

(3) More generally, if V_1 and V_2 are G-representations, then $\mathrm{Hom}(V_1, V_2)$ is as well, via

$$\rho_{\mathrm{Hom}(V_1,V_2)}(g)(F) = \rho_{V_2}(g) \circ F \circ \rho_{V_1}(g^{-1}).$$

Here (2) is a special case where we view $\mathbb{C}$ as the trivial representation of G. With this definition, the fixed subspace $(\mathrm{Hom}(V_1, V_2))^G = \mathrm{Hom}^G(V_1, V_2)$, the space of G-morphisms from V_1 to V_2.

(4) Finally, if V_1 and V_2 are G-representations, then $V_1 \otimes V_2$ is as well, via

$$\rho_{V_1 \otimes V_2} = \rho_{V_1} \otimes \rho_{V_2}.$$

It is easy to check that the "natural" isomorphisms $V \cong V^{**}$ and $\mathrm{Hom}(V, W) \cong V^* \otimes W$ are all G-isomorphisms. However, in general V^* is **not** G-isomorphic to V.

4 Irreducible representations

Definition 4.1. A G-representation V is *irreducible* if $V \neq \{0\}$, and the only G-invariant subspaces of V are V and $\{0\}$. A G-representation V is *reducible* if it is not irreducible.

Example 4.2. (1) If $\dim V = 1$, then V is irreducible.

(2) A two dimensional representation is reducible $\iff$ there exists a common nonzero eigenvector for G.

(3) The standard representation $\mathbb{C}^n$ of S^n is not irreducible, since it has the two subspaces W_1, W_2. However, W_1 is irreducible because $\dim W_1 = 1$, and we will see that W_2 is also irreducible for every $n \geq 2$.

Lemma 4.3. *If $V \neq \{0\}$, then there exists a G-invariant subspace $W \neq \{0\}$ of V which is an irreducible G-representation.*

Proof. The proof is by complete induction on $\dim V$. If $\dim V = 1$, then V is irreducible and we can take $W = V$. For the inductive step, suppose that the result has been proved for all representations of degree less than n. Let V be a representation of degree n. If V is irreducible, then as before we can take $W = V$. If V is not irreducible, then there exists a G-invariant subspace V' of V with $1 \leq \deg V' < \deg V = n$. By the inductive hypothesis, there exists a G-invariant subspace $W \neq \{0\}$ of V' which is an irreducible G-representation. Then W is a nonzero G-invariant subspace of V which is an irreducible G-representation. This completes the inductive step and hence the proof. $\square$

Lemma 4.4. *Let $F\colon V \to W$ be a morphism of G-representations.*

(i) *If V is irreducible, then F is either 0 or injective.*

(ii) *If W is irreducible, then F is either 0 or surjective.*

(iii) *If both V and W are irreducible, then F is either 0 or an isomorphism.*

Proof. (i) We have seen that $\operatorname{Ker} F$ is a G-invariant subspace of V. Hence either $\operatorname{Ker} F\{0\}$ or $\operatorname{Ker} F = V$. In the first case, F is injective, and in the second case $F = 0$.

(ii) Similarly, $\operatorname{Im} F$ is a G-invariant subspace of W. Hence either $\operatorname{Im} F = \{0\}$ or $\operatorname{Im} F = W$. In the first case, $F = 0$, and in the second case F is surjective.

(iii) This follows from (i) and (ii). $\square$

Proposition 4.5 (Schur's lemma)**.** *Let V be an irreducible G-representation and let $F \in \operatorname{Hom}^G(V, V)$. Then there exists a $t \in \mathbb{C}$ such that $F = t\operatorname{Id}$. Hence, if V and W are two irreducible G-representations, then either V is not isomorphic to W and $\operatorname{Hom}^G(V, W) = 0$, or V is isomorphic to W and $\dim \operatorname{Hom}^G(V, W) = 1$.*

Proof. We have seen that every element of $\operatorname{Hom}^G(V, V)$ is either 0 or an isomorphism. Let $F \in \operatorname{Hom}^G(V, V)$. Then there exists a (nonzero) eigenvector $v \in V$, i.e. a nonzero $v \in V$ such that there exists a $t \in \mathbb{C}$ with $F(v) = tv$. Thus, the G-morphism $F - t\operatorname{Id}$ is not invertible, since $v \in \operatorname{Ker}(F - t\operatorname{Id})$ and $v \neq 0$. It follows that $F - t\operatorname{Id} = 0$. Hence $F = t\operatorname{Id}$.

For the proof of the last statement, if V and W are not isomorphic, then, by (iii) of Lemma 4.4, $\operatorname{Hom}^G(V, W) = 0$. If V is G-isomorphic to W, we may as well assume that $V = W$, and then the argument above shows that $\operatorname{Hom}^G(V, V) \cong \mathbb{C}$, hence has dimension one. $\square$

Remark 4.6. The proof above used the fact that the characteristic polynomial of F had a root, which follows since every nonzero polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$. In the terminology of Modern Algebra II, $\mathbb{C}$ is *algebraically closed*. In general, for a field k, we have defined k-representations and can speak of a k-representation $V \neq \{0\}$ as being *k-irreducible*, i.e. there are no G-invariant k-subspaces of V except for $\{0\}$ and V. The proof of Schur's lemma then shows that $\operatorname{Hom}^G(V, V)$ is a division ring containing k as a subfield. There exist examples of $\mathbb{R}$-irreducible $\mathbb{R}$-representations V for which the ring $\operatorname{Hom}^G(V, V)$ is isomorphic to $\mathbb{C}$, and examples where $\operatorname{Hom}^G(V, V)$ is isomorphic to $\mathbb{H}$.

We turn next to the construction of G-invariant projections. Here, the methods will only work in the case of a **finite group** (although we shall make some remarks about other cases later).

Proposition 4.7. *Let G be a finite group. Suppose that V is a G-representation, and define $p\colon V \to V$*

$$p(v) = \frac{1}{\#(G)} \sum_{g \in G} \rho_V(g)(v).$$

Then p is a G-morphism with $\operatorname{Im} p = V^G$ and $p(v) = v$ for all $v \in V^G$, i.e. p is a G-invariant projection from V to V^G. Hence, as G-representations, $V \cong V^G \oplus W$, where $W = \operatorname{Ker} p$ is a G-invariant subspace.

Proof. First, if $v \in V^G$, then by definition $\rho_V(g)(v) = v$ for all $g \in G$. Thus

$$p(v) = \frac{1}{\#(G)} \sum_{g \in G} \rho_V(g)(v) = \frac{1}{\#(G)} \sum_{g \in G} v = \frac{1}{\#(G)}(\#(G)v) = v.$$

In particular, $V^G \subseteq \operatorname{Im} p$.

Next, if $h \in G$ and $v \in V$, then

$$\begin{aligned} \rho_V(h)p(v) = \rho_V(h)\left(\frac{1}{\#(G)} \sum_{g \in G} \rho_V(g)(v)\right) &= \frac{1}{\#(G)} \sum_{g \in G} \rho_V(h) \circ \rho_V(g)(v) \\ &= \frac{1}{\#(G)} \sum_{g \in G} \rho_V(hg)(v). \end{aligned}$$

But, as g runs through G, hg also runs through the elements of G. Hence

$$\frac{1}{\#(G)} \sum_{g \in G} \rho_V(hg)(v) = \frac{1}{\#(G)} \sum_{g \in G} \rho_V(g)(v) = p(v).$$

Thus, for all $v \in V$ and $h \in G$, $\rho_V(h)p(v) = p(v)$. Hence $\operatorname{Im} p \subseteq V^G$, and so $\operatorname{Im} p = V^G$ since we have already showed that $V^G \subseteq \operatorname{Im} p$. It follows that $V \cong W \oplus V^G$ (internal direct sum), where $W = \operatorname{Ker} p$.

Next we show that p is a G-morphism. This is a very similar argument to the proof above that $\operatorname{Im} p \subseteq V^G$. Since the G-action on $\operatorname{Im} p = V^G$ is trivial, it suffices to show that $p \circ \rho_V(h) = p$ for all $h \in G$. But

$$p \circ \rho_V(h) = \frac{1}{\#(G)} \sum_{g \in G} \rho_V(g) \circ \rho_V(h) = \frac{1}{\#(G)} \sum_{g \in G} \rho_V(gh).$$

As before, as g runs through G, gh also runs through the elements of G. Thus

$$\frac{1}{\#(G)}\sum_{g\in G}\rho_V(gh) = \frac{1}{\#(G)}\sum_{g\in G}\rho_V(g) = p,$$

so that $p\circ\rho_V(h) = p = \rho_V(h)\circ p$ for all $h\in G$.

Finally, since p is a G-morphism, $W = \operatorname{Ker} p$ is a G-invariant subspace of V. We have seen that, as G-representations, $V \cong W \oplus V^G$. $\square$

Remark 4.8. If $V^G = \{0\}$, then Proposition 4.7 tells us that, for all $v \in V$,

$$\sum_{g\in G}\rho_V(g)(v) = 0.$$

Definition 4.9. V is *decomposable* if there exist two nonzero G-invariant subspaces W_1, W_2 of V such that $V \cong W_1 \oplus W_2$. V is *completely reducible* if $V \neq 0$ and there exist irreducible G-representations $V_1, \ldots, V_k$ such that $V \cong V_1 \oplus \cdots \oplus V_k$. For example, an irreducible representation is completely reducible (take $k = 1$ above). Clearly, if $V \cong W_1 \oplus W_2$ and W_1, W_2 are completely reducible, then V is completely reducible as well.

Theorem 4.10 (Maschke's theorem)**.** *If G is finite and W is a G-invariant subspace of G, then there exists a G-invariant subspace W' of V such that $V \cong W \oplus W'$.*

Proof. We will find a G-morphism $p\colon V \to V$ such that $\operatorname{Im} p \subseteq W$ and $p(w) = w$ for all $w \in W$. Setting $W' = \operatorname{Ker} p$, it then follows that W' is also G-invariant, and that V is the internal direct sum of W and W'. Then Lemma 3.1 implies that the sum map $W \oplus W' \to V$ is an isomorphism of G-representations.

To find p, begin by choosing an arbitrary linear map $p_0\colon V \to V$ such that $\operatorname{Im} p_0 \subseteq W$ and $p_0(w) = w$ for all $w \in W$. For example, choose a basis $w_1, \ldots, w_a, w_{a+1}, \ldots, w_n$ of V such that $w_1, \ldots, w_a$ is a basis of W and define p_0 by defining it on the basis vectors w_i by defining

$$p_0(w_i) = \begin{cases} w_i, & \text{if } i \leq a;\\ 0, & \text{if } i > a.\end{cases}$$

Then set

$$p = \frac{1}{\#(G)}\sum_{g\in G}\rho_V(g)\circ p_0\circ\rho_V(g)^{-1} = \frac{1}{\#(G)}\sum_{g\in G}\rho_{\mathrm{Hom}(V,V)}(g)(p_0),$$

viewing p_0 as an element of the G-representation $\mathrm{Hom}(V, V)$. By Proposition 4.7, $p \in \mathrm{Hom}^G(V, V)$, so that p is a G-morphism. Since $\mathrm{Im}\, p_0 \subseteq W$ and W is G-invariant, $\mathrm{Im}\, p \subseteq W$. Finally, if $w \in W$, then $\rho_V(g)^{-1}(w) \in W$ as well, again since W is G-invariant. Then $p_0(\rho_V(g)^{-1}(w)) = \rho_V(g)^{-1}(w)$ by construction, and so

$$\begin{aligned} p(w) &= \frac{1}{\#(G)} \sum_{g\in G} \rho_V(g)(p_0(\rho_V(g)^{-1}(w))) = \frac{1}{\#(G)} \sum_{g\in G} \rho_V(g)(\rho_V(g)^{-1}(w)) \\ &= \frac{1}{\#(G)} \sum_{g\in G} w = w. \end{aligned}$$

Thus p has the desired properties. $\square$

Corollary 4.11. *If G is finite, then every nonzero G-representation V is completely reducible.*

Proof. The proof is by complete induction on the degree of a G-representation. If $\dim V = 1$, then V is irreducible and so (as we have already noted) it is completely reducible. Now suppose that the corollary has been proved for all representations of degree less than n. If V is a representation of degree n, first suppose that V is irreducible. Then as above V is completely reducible. Otherwise, V is reducible, so there exists a G-invariant subspace W of V with $0 < \deg W < n$. By Maschke's theorem, V is G-isomorphic to $W \oplus W'$, where $\deg W' = n - \deg W$, and hence $0 < \deg W' < n$ as well. By the inductive hypothesis, W and W' are completely reducible. Thus, $V \cong W \oplus W'$ is completely reducible as well. $\square$

Corollary 4.12. *Suppose that G is a finite abelian group. Then every nonzero G-representation V is a direct sum of one dimensional representations. Equivalently, there is a basis of V consisting of common eigenvectors for G.*

Proof. It is clearly enough to prove that, if G is a finite abelian group, then every irreducible representation of G is one-dimensional. Let V be an irreducible G-representation. In particular $V \neq 0$. By Schur's lemma, $\mathrm{Hom}^G(V, V) = \mathbb{C} \cdot \mathrm{Id}$. On the other hand, we have seen that, if G is abelian, then, for every $g \in G$, $\rho_V(g) \in \mathrm{Hom}^G(V, V)$, and hence $\rho_V(g) = \lambda(g)\,\mathrm{Id}$ for some $\lambda(g) \in \mathbb{C}^*$. Thus, choosing some nonzero $v \in V$, $\rho_V(g) = \lambda(g)v$ for every $g \in G$. It follows that the one-dimensional subspace $\mathbb{C} \cdot v = \mathrm{span}\{v\}$ is a G-invariant, nonzero subspace of V. Since V is irreducible, $V = \mathbb{C} \cdot v$ and hence V is one-dimensional. $\square$

Corollary 4.13. *If $A \in GL(n, \mathbb{C})$ is a matrix of finite order d, then A is diagonalizable and its eigenvalues are d^{th} roots of unity.*

Proof. If A has order d, then A defines a representation of $\mathbb{Z}/d\mathbb{Z}$ on $\mathbb{C}^n$ by: $\rho(k) = A^k$. Then by the previous corollary, $\mathbb{C}^n$ is a direct sum of eigenspaces for A. Since $A^d = \mathrm{Id}$, it is clear that all of the eigenvalues of A are d^{th} roots of unity. $\square$

Corollary 4.14. *If G is a finite group and V is a G-representation, then, for all $g \in G$, the linear map $\rho_V(g)\colon V \to V$ is diagonalizable, and its eigenvalues are d^{th} roots of unity, where d divides $\#(G)$.*

Proof. Every element g of G has finite order dividing $\#(G)$, by Lagrange's theorem. Moreover, $\rho_V(g)$ has finite order dividing the order of g, and hence dividing $\#(G)$. Then apply the previous corollary. $\square$

Remark 4.15. For a not necessarily finite group G, a G-representation V is *unitary* if there exists a positive definite Hermitian inner product H which is G-invariant, i.e. for which $H(\rho_V(g)v, \rho_V(g)w) = H(v, w)$, for all $v, w \in V$ and $g \in G$. If V is unitary, then there exists a basis of V for which $\rho_V(g)$ is unitary for all $g \in G$, i.e. there exists a choice of basis for which ρ_V is a homomorphism to $U(n)$. Every unitary representation satisfies Maschke's theorem, because if $W \subseteq V$ is G-invariant, then $W^\perp$ is also G-invariant and $V \cong W \oplus W^\perp$. If G is finite, then there always exists a G-invariant positive definite Hermitian inner product H: start with an arbitrary positive definite Hermitian inner product H_0, and set

$$H(v, w) = \sum_{g \in G} H_0(\rho_V(g)v, \rho_V(g)w).$$

Then H is G-invariant.

Example 4.16. We have seen that every $A \in GL(n, \mathbb{C})$ defines a representation of $\mathbb{Z}$ on $\mathbb{C}^n$, via $\rho(n) = A^n$. In particular, defines a $\mathbb{Z}$-representation on $\mathbb{C}^2$ by taking $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and hence $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Note that $A^n e_1 = e_1$ and $A^n e_2 = e_2 + ne_1$. Thus $\mathbb{C} \cdot e_1$ is a $\mathbb{Z}$-invariant subspace. In fact, it is the unique $\mathbb{Z}$-invariant subspace: if $W \neq \{0\}, \mathbb{C}^2$ is an invariant subspace, then $\dim W = 1$ and $W = \mathbb{C} \cdot w$ where w is a nonzero eigenvector for A and hence A^n. But A has a unique nonzero eigenvector up to a scalar, namely e_1. It follows that the $\mathbb{Z}$-representation $\mathbb{C}^2$ is not completely reducible. Hence there is no $\mathbb{Z}$-invariant positive definite Hermitian inner product on $\mathbb{C}^2$.

From now on, G will always denote a finite group.