Induced representations

1 Definition of induced representations

Induced representations are a new method for constructing representations of a finite group G, starting with a subgroup $H \leq G$ and a representation ρ_W of H, i.e. a homomorphism $\rho_W : H \to \operatorname{Aut} W$. The result is a new representation $\operatorname{Ind}_H^G W = V$ of G, of dimension $(G : H) \dim W$. However, even if W is irreducible, $\operatorname{Ind}_H^G W$ need not be irreducible.

The simplest definition involves tensor products over non-commutative rings. An H-representation W is the same thing as a left $\mathbb{C}[H]$-module. Moreover, $\mathbb{C}[G]$ is either a left or a right $\mathbb{C}[H]$-module, depending on which side we choose to multiply. Viewing $\mathbb{C}[G]$ as a right $\mathbb{C}[H]$-module, we can form the tensor product $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$. It loses the action of $\mathbb{C}[H]$ but gains an action of $\mathbb{C}[G]$ because $\mathbb{C}[G]$ acts on itself by left multiplication, and this commutes with right multiplication by $\mathbb{C}[H]$. Thus $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ is a left $\mathbb{C}[G]$-module, and thus defines a G-representation which we denote by $\operatorname{Ind}_H^G W$.

We will give a concrete description of this construction as follows:

Definition 1.1. Let G be a finite group, let H be a subgroup of G, and let $\rho_W : H \to \operatorname{Aut} W$ be an H-representation. Then we define $\operatorname{Ind}_H^G W$ to be the vector space of all functions $F : G \to W$ such that, for all $g \in G$ and $h \in H$,

$$F(gh) = \rho_W(h)^{-1}(F(g)).$$

Lemma 1.2. *With notation as above, $\operatorname{Ind}_H^G W$ is a vector subspace of the vector space of all functions from G to W under pointwise addition and scalar multiplication. It is a G-representation under the action of g given by*

$$\rho_{\operatorname{Ind}_H^G W}(g)(F)(x) = F(g^{-1}x).$$

Proof. The content of the first statement is that, if F_1 and F_2 satisfy the condition of Definition 1.1, then so do $F_1 + F_2$ and tF_1, $t \in \mathbb{C}$. The content of the second statement is that, for all $g_1, g_2 \in G$,

$$\rho_{\operatorname{Ind}_H^G W}(g_1)\rho_{\operatorname{Ind}_H^G W}(g_2)(F) = \rho_{\operatorname{Ind}_H^G W}(g_1 g_2)(F).$$

These are straightforward calculations. $\square$

Example 1.3. (1) If $H = \{1\}$ is the trivial subgroup of G and $W \cong \mathbb{C}$ is the trivial representation, then the condition $F(gh) = \rho_W(h)^{-1}(F(g))$ is trivially satisfied for all $g \in G$, since the only element of H is 1 and $\rho_W(1)^{-1} = \mathrm{Id}$. Thus, as a vector space, $\operatorname{Ind}_H^G W = L^2(G)$, the vector space of all functions from G to $\mathbb{C}$. Moreover, the action of G on $L^2(G)$ is the usual action, precomposition with g^{-1}. Thus

$$\operatorname{Ind}_{\{1\}}^G \mathbb{C} = L^2(G) = \mathbb{C}[G],$$

the regular representation.

(2) More generally, suppose that H is an arbitrary subgroup of G but that again $W = \mathbb{C}$ is the trivial representation of H. Then $\operatorname{Ind}_H^G \mathbb{C}$ is the set of all functions $F\colon G \to \mathbb{C}$ such that, for all $g \in G$ and $h \in H$, $F(gh) = \rho_W(h)^{-1}(F(g)) = \operatorname{Id} F(g) = F(g)$. In other words, $\operatorname{Ind}_H^G \mathbb{C}$ is the set of all functions $F\colon G \to \mathbb{C}$ which are constant on the cosets gH, for every $g \in G$. Such a function is the same thing as a function $f\colon G/H \to \mathbb{C}$. Moreover, it is easy to check that the G-action is given by

$$\rho_{\operatorname{Ind}_H^G \mathbb{C}}(g)(f)(xH) = f(g^{-1}xH).$$

This is the usual action of G on the permutation representation $\mathbb{C}[G/H]$. In particular, we see that $\dim \operatorname{Ind}_H^G \mathbb{C} = (G : H)$ in this case.

(3) Suppose that $H = G$ and that V is a G-representation. Then $\operatorname{Ind}_G^G V$ is the set of functions $F\colon G \to V$ such that, for all $g_1, g_2 \in G$,

$$F(g_1 g_2) = \rho_V(g_2)^{-1} F(g_1).$$

In particular, taking $g_1 = 1$ and $g_2 = g$ gives

$$F(g) = F(1 \cdot g) = \rho_V(g)^{-1} F(1).$$

In particular, the vector $v = F(1)$ determines F. Conversely, suppose that $v \in V$ and define the function $F_v\colon G \to V$ by:

$$F_v(g) = \rho_V(g)^{-1}(v).$$

Then we see that $F_v \in \operatorname{Ind}_G^G V$ because

$$F_v(g_1g_2) = \rho_V(g_1g_2)^{-1}(v) = \rho_V(g_2)^{-1}\rho_V(g_1)^{-1}(v) = \rho_V(g_2)^{-1}F_v(g_1).$$

Thus, if we define functions $A\colon \operatorname{Ind}_G^G V \to V$ and $B\colon V \to \operatorname{Ind}_G^G V$ via

$$\begin{aligned} A(F) &= F(1); \\ B(v) &= F_v, \end{aligned}$$

then it is easy to see that A and B are inverse linear maps, and they are G-isomorphisms because

$$\begin{aligned} A(\rho_V(g)(F)) &= \rho_V(g)(F)(1) = F(g^{-1} \cdot 1) = F(g^{-1}) \\ &= \rho_V(g)F(1) = \rho_V(g)(A(F)). \end{aligned}$$

Thus, as G-representations, $\operatorname{Ind}_G^G V \cong V$.

For a better understanding of $\operatorname{Ind}_H^G W$ in general, we will need an explicit way to compute it and especially to compute its character $\chi_{\operatorname{Ind}_H^G}$. We fix the following notation: $x_1, \dots, x_k$ are representatives for the set of left cosets G/H. In other words, every element of G is in exactly one left coset x_iH. In particular $k = (G : H)$. By convention, we will always take $x_1 = 1$. Given $w \in W$ and i, $1 \leq i \leq k$, we define a function $F_{i,w}\colon G \to W$ by the formula

$$F_{i,w}(g) = \begin{cases} \rho_W(h)^{-1}(w), & \text{if } g = x_ih \in x_iH; \\ 0, & \text{otherwise.} \end{cases}$$

Note that, in particular, $F_{i,w}(x_i) = w$.

Lemma 1.4. *With $F_{i,w}$ defined as above,*

(i) $F_{i,w} \in \operatorname{Ind}_H^G W$.

(ii) *Given $w_1, w_2 \in W$ and $t \in \mathbb{C}$, $F_{i,w_1} + F_{i,w_2} = F_{i,w_1+w_2}$ and $F_{i,tw_1} = tF_{i,w_1}$.*

(iii) *If we define $W^{(i)} = \{F_{i,w} : w \in W\}$, then $W^{(i)}$ is a vector subspace of $\operatorname{Ind}_H^G W$, isomorphic as a vector space to W, and moreover*

$$W^{(i)} = \{F \in \operatorname{Ind}_H^G W : F(g) = 0 \text{ if } g \neq x_iH\}.$$

(iv) *For all $F \in \operatorname{Ind}_H^G W$, define $w_i = F(x_i)$. Then*

$$F = \sum_{i=1}^{k} F_{i,w_i}.$$

(v) *As vector spaces,* $\operatorname{Ind}_H^G W \cong \bigoplus_{i=1}^k W^{(i)}$. *In particular,*

$$\dim \operatorname{Ind}_H^G W = k \dim W = (G : H) \dim W.$$

Proof. (i) Let $k \in H$. If $g = x_i h \in x_i H$, then $gk = x_i hk \in x_i H$ and

$$\begin{aligned} F_{i,w}(gk) = F_{i,w}(x_i hk) &= \rho_W(hk)^{-1}(w) = \rho_W(k)^{-1}\rho_W(h)^{-1}(w) \\ &= \rho_W(k)^{-1} F_{i,w}(g). \end{aligned}$$

If $x_i h \notin x_i H$, then $x_i hk \notin x_i H$ as well, and hence

$$F_{i,w}(gk) = 0 = \rho_W(k)^{-1}(0) = \rho_W(k)^{-1} F_{i,w}(g).$$

Thus, in all cases, $F_{i,w}(gk) = \rho_W(k)^{-1} F_{i,w}(g)$ and so $F_{i,w} \in \operatorname{Ind}_H^G W$.
(ii) This is clear because $\rho_W(h)^{-1}$ is linear.
(iii) It follows from (ii) that $W^{(i)}$ is a vector subspace of $\operatorname{Ind}_H^G W$. Clearly, the map $w \mapsto F_{i,w}$ is linear, with inverse $F \mapsto F(x_i)$. Thus $W^{(i)} \cong W$ as vector spaces. By definition

$$W^{(i)} \subseteq \{F \in \operatorname{Ind}_H^G W : F(g) = 0 \text{ if } g \neq x_i H\}.$$

Conversely, suppose that $F \in \operatorname{Ind}_H^G W$ and that $F(g) = 0$ if $g \neq x_i H$. Define $w = F(x_i)$. By the definition of $\operatorname{Ind}_H^G W$, if $g = x_i h$, then

$$F(g) = F(x_i h) = \rho_W(h)^{-1} F(x_i) = \rho_W(h)^{-1}(w).$$

Thus $F = F_{i,w}$, and hence

$$\{F \in \operatorname{Ind}_H^G W : F(g) = 0 \text{ if } g \neq x_i H\} \subseteq W^{(i)}.$$

It follows that

$$W^{(i)} = \{F \in \operatorname{Ind}_H^G W : F(g) = 0 \text{ if } g \neq x_i H\}.$$

(iv) Given $F \in \operatorname{Ind}_H^G W$ and $w_i = F(x_i)$, we must show that $F(g) = \sum_{i=1}^k F_{i,w_i}(g)$ for all $g \in G$. There is a unique i, $1 \le i \le k$, such that $g \in x_i H$ and hence such that $g = x_i h$ for some $h \in H$. Thus $F(g) = F(x_i h) = \rho_W(h)^{-1} F(x_i) = \rho_W(h)^{-1}(w_i)$. On the other hand, for $j \neq i$, $F_{j,w_j}(g) = 0$ and $F_{i,w_i}(g) = F_{i,w_i}(x_i h) = \rho_W(h)^{-1}(w_i)$. Thus $F(g) = \sum_{i=1}^k F_{i,w_i}(g)$.
(v) The above argument shows that every element F of $\operatorname{Ind}_H^G W$ can be written as a sum of elements in the $W^{(i)}$, $1 \le i \le k$. In fact, it can be uniquely so written, since if also $F = \sum_{i=1}^k F_i'$ with $F_i' \in W^{(i)}$, then it is easy to see that necessarily $F_i'(x_i h) = \rho_W(h)^{-1} F(x_i)$ and that $F_i'(g) = 0$ if $g \notin x_i H$. Thus $F_i' = F_{i,w_i}$. The statement about $\dim \operatorname{Ind}_H^G W$ then follows since $\dim W^{(i)} = \dim W$ for all i. $\square$

Using the above lemma, we can give an explicit description for the action of $g \in G$ on $\operatorname{Ind}_H^G W$. Given $F \in \operatorname{Ind}_H^G W$, let $F = \sum_{i=1}^k F_{i,w_i}$. It suffices to describe $\rho_{\operatorname{Ind}_H^G W}(F_{i,w_i})$. Given $g \in G$, gx_i is in a unique left coset x_jH of H, and hence we can write

$$gx_i = x_j h_i(g)$$

for a unique j and $h_i(g)$, depending on i and g. Note that j does not have to equal i, even if $g \in H$. However, if H is normal and $g \in H$, then $gx_i = x_i h_i(g)$ for some $h_i(g) \in H$.

Claim 1.5. *With notation as above,*

$$\boxed{\rho_{\operatorname{Ind}_H^G W}(g)(F_{i,w}) = F_{j,\rho_W(h_i(g))(w)}}$$

Proof. It suffices to compute $\rho_{\operatorname{Ind}_H^G W}(g)(F_{i,w})(x_\ell)$ for every ℓ. By definition, $\rho_{\operatorname{Ind}_H^G W}(g)(F_{i,w})(x_\ell) = F_{i,w}(g^{-1}x_\ell)$. Now $F_{i,w}(g^{-1}x_\ell) = 0$ if $g^{-1}x_\ell \notin x_iH$, i.e. if $x_\ell \notin gx_iH = x_jH$, or equivalently if $\ell \neq j$. If $\ell = j$, then

$$g^{-1}x_j = x_i h_i(g)^{-1},$$

since $gx_i = x_j h_i(g)$, and hence

$$\begin{aligned} F_{i,w}(g^{-1}x_j) &= F_{i,w}(x_i h_i(g)^{-1}) = \rho_W(h_i(g))F_{i,w}(x_i) \\ &= \rho_W(h_i(g))(w) = F_{j,\rho_W(h_i(g))(w)}(x_j). \end{aligned}$$

Thus $\rho_{\operatorname{Ind}_H^G W}(g)(F_{i,w})(x_\ell) = F_{j,\rho_W(h_i(g))(w)}(x_\ell)$ for every ℓ, and hence

$$\rho_{\operatorname{Ind}_H^G W}(g)(F_{i,w}) = F_{j,\rho_W(h_i(g))(w)}.$$

□

Let us make some additional remarks about $\operatorname{Ind}_H^G W$ in general. Recall that, with our conventions, $x_1 = 1$. Thus, for all $h \in H$, $hx_1 = h = x_1h$, so that $h_1(h) = h$. Hence

$$\rho_{\operatorname{Ind}_H^G W}(h)(F_{1,w}) = F_{1,\rho_W(h)(w)}.$$

This says that the subspace $W^{(1)}$ is an H-invariant subspace of $\operatorname{Ind}_H^G W$ and it is H-isomorphic to W. Also, since $x_j \cdot x_1 = x_j$, we have $h_1(x_j) = 1$. Thus

$$\rho_{\operatorname{Ind}_H^G W}(x_j)(F_{1,w}) = F_{j,w}.$$

This is often written symbolically as

$$\operatorname{Ind}_H^G W \cong \bigoplus_{i=1}^k x_i \cdot W.$$

Finally, given a general element $x_jh \in x_jH$, since $(x_jh)x_1 = x_jh$, $h_1(x_jh) = h$ and

$$\rho_{\operatorname{Ind}_H^G W}(x_jh)(F_{1,w}) = F_{j,\rho_W(h)(w)}.$$

For example, suppose that W is a 1-dimensional representation of H, with basis vector e and corresponding to the homomorphism $\lambda\colon H \to \mathbb{C}^*$. Set $f_i = F_{i,e}$. Then $f_1, \dots, f_k$ is a basis of $\operatorname{Ind}_H^G W$, and

$$\rho_{\operatorname{Ind}_H^G W}(g)(f_i) = \lambda(h_i(g))f_j.$$

This says that G acts on the basis $f_1, \dots f_k$ by a combination of the permutation representation and multiplication by scalars.

Example 1.6. Let D_n be the dihedral group of order $2n$, which we can view as generated by two elements α and τ, with $\alpha^n = 1$, $\tau^2 = 1$, and $\tau\alpha\tau^{-1} = \tau\alpha\tau = \alpha^{-1}$. (We previously called the generator of the rotation subgroup ρ but want to avoid confusion with the letter used to denote a representation.) Thus every element of D_n is uniquely written either as α^k or as $\tau\alpha^k$, with $0 \le k \le n-1$, and the cyclic subgroup $H = \langle\alpha\rangle$ has index two in D_n. Let $W_a = \mathbb{C} \cdot e$ be the 1-dimensional representation of H corresponding to the homomorphism $\lambda_a(\alpha^k) = e^{2\pi iak/n}$, i.e. e is a basis vector for W_a and $\rho_{W_a}(\alpha^k)(e) = e^{2\pi iak/n}e$.

Then D_n/H has two elements, and we can take as a set of representatives $x_1 = 1$ and $x_2 = \tau$. For $g = \alpha^k$, $\alpha^k \cdot x_1 = \alpha^k = x_1\alpha^k$ and so $h_1(\alpha^k) = \alpha^k$. Also,

$$\alpha^k \cdot x_2 = \alpha^k \cdot \tau = \tau \cdot \alpha^{-k} = x_2 \cdot \alpha^{-k},$$

and so $h_2(\alpha^k) = \alpha^{-k}$. It follows that (with $f_i = F_{i,e}$ as before and $i = 1, 2$)

$$\rho_{\operatorname{Ind}_H^{D_n} W_a}(\alpha^k)(f_1) = \lambda_a(\alpha^k)f_1 = e^{2\pi iak/n}f_1;$$
$$\rho_{\operatorname{Ind}_H^{D_n} W_a}(\alpha^k)(f_2) = \lambda_a(\alpha^{-k})f_2 = e^{-2\pi iak/n}f_2.$$

Since $\tau x_1 = \tau = x_2$, $h_1(\tau) = 1$, and $\tau x_2 = \tau^2 = 1 = x_1$, so $h_2(\tau) = 1$ as well. Thus

$$\rho_{\operatorname{Ind}_H^{D_n} W_a}(\tau)(f_1) = f_2;$$
$$\rho_{\operatorname{Ind}_H^{D_n} W_a}(\tau)(f_2) = f_1.$$

If we write these out as 2×2 matrices using the basis $\{f_1, f_2\}$, then

$$\rho_{\mathrm{Ind}_H^{D_n} W_a}(\alpha^k) = \begin{pmatrix} e^{2\pi iak/n} & 0 \\ 0 & e^{-2\pi iak/n} \end{pmatrix}; \qquad \rho_{\mathrm{Ind}_H^{D_n} W_a}(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$\rho_{\mathrm{Ind}_H^{D_n} W_a}(\tau\alpha^k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{2\pi iak/n} & 0 \\ 0 & e^{-2\pi iak/n} \end{pmatrix} = \begin{pmatrix} 0 & e^{-2\pi iak/n} \\ e^{2\pi iak/n} & 0 \end{pmatrix}.$$

This gives the formula for the character:

$$\chi_{\mathrm{Ind}_H^{D_n} W_a}(\alpha^k) = 2\cos\frac{2\pi iak}{n};$$
$$\chi_{\mathrm{Ind}_H^{D_n} W_a}(\tau\alpha^k) = 0.$$

We leave it as a homework problem to show that every 2-dimensional representation of D_n is of the form $\mathrm{Ind}_H^{D_n} W_a$ for some a, and to decide when two such are isomorphic.

2 The character of an induced representation

Our goal in this section is to generalize the computation in Example 1.6 to give a formula for the character of an induced representation in general. We keep the previous notation. In particular, $\mathrm{Ind}_H^G W \cong \bigoplus_{i=1}^k W^{(i)}$, where $W^{(i)} \cong W$ is the span of the functions $F_{i,w}, w \in W$. Given $g \in G$, with $gx_i = x_j h_i(g)$, we want to compute $\mathrm{Tr}\, \rho_{\mathrm{Ind}_H^G W}(g) = \chi_{\mathrm{Ind}_H^G W}(g)$. We have seen that

$$\rho_{\mathrm{Ind}_H^G W}(g)(F_{i,w}) = F_{j,\rho_W(h_i(g))(w)}.$$

To compute the trace, the only nonzero contributions will come from those i such that $j = i$, i.e. such that $gx_i = x_i h_i(g)$, or equivalently such that $x_i^{-1} g x_i = h_i(g) \in H$. For such i, $\rho_{\mathrm{Ind}_H^G W}(g)$ induces a linear map $W^{(i)} \to W^{(i)}$, and under the identification of $W^{(i)}$ with W, by identifying $F_{i,w}$ with $w \in W$, we see that the action of $\rho_{\mathrm{Ind}_H^G W}(g)$ restricted to $W^{(i)}$ is identified with

$$\rho_W(h_i(g)) = \rho_W(x_i^{-1} g x_i).$$

Thus, summing over all possible i, we have:

$$\chi_{\mathrm{Ind}_H^G W}(g) = \mathrm{Tr}\, \rho_{\mathrm{Ind}_H^G W}(g) = \sum_{\substack{i \text{ such that} \\ x_i^{-1} g x_i \in H}} \mathrm{Tr}\, \rho_W(x_i^{-1} g x_i) = \sum_{\substack{i \text{ such that} \\ x_i^{-1} g x_i \in H}} \chi_W(x_i^{-1} g x_i).$$

Note that we don't expect that $\chi_W(x_i^{-1}gx_i) = \chi_W(g)$, since $\chi_W(g)$ is not even defined for $g \notin H$, and in any case $x_i \notin H$ for general i, so the conjugation is not by elements of H.

We can eliminate the choice of the x_i from the above formula. Here x_i is one particular representative for the coset x_iH. Any other representative $x = x_ih$ will still have the property that $g(x_ih) \in x_iH$ and moreover

$$x^{-1}gx = h^{-1}x_i^{-1}gx_ih = h^{-1}h_i(g)h.$$

Thus, as χ_W is a class function **on** H,

$$\chi_W(x^{-1}gx) = \chi_W(h^{-1}h_i(g)h) = \chi_W(h_i(g)) = \chi_W(x_i^{-1}gx_i).$$

Thus, if we sum $\chi_W(x^{-1}gx)$ over the $x \in x_iH$, we get $\#(H)\chi_W(x_i^{-1}gx_i)$. Summing over all $x \in G$ such that $x^{-1}Hx = H$, we get the following formula for $\chi_{\mathrm{Ind}_H^G W}(g)$:

$$\boxed{\chi_{\mathrm{Ind}_H^G W}(g) = \frac{1}{\#(H)} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi_W(x^{-1}gx)}$$

We can interpret this formula as follows.

Definition 2.1. Let f be a class function on H, and define a function $\tilde{f}$ on G by the formula

$$\tilde{f}(g) = \begin{cases} f(g), & \text{if } g \in H; \\ 0, & \text{if } g \notin H. \end{cases}$$

Note that $\tilde{f}$ is not necessarily a class function. Then define $\mathrm{Ind}_H^G f \colon G \to \mathbb{C}$ by the formula:

$$\mathrm{Ind}_H^G f(g) = \frac{1}{\#(H)} \sum_{x \in G} \tilde{f}(x^{-1}gx).$$

This is always a class function since we sum the values of $\tilde{f}$ over the conjugates of g.

With this notation, our formula for $\chi_{\mathrm{Ind}_H^G W}$ reads:

$$\boxed{\chi_{\mathrm{Ind}_H^G W} = \mathrm{Ind}_H^G \chi_W}$$

Of course, there is a similar formula for $\mathrm{Res}_H^G \chi_V$, where V is a G-representation, namely

$$\chi_{\mathrm{Res}_H^G V} = \mathrm{Res}_H^G \chi_V.$$

But in this case the proof is obvious from the definitions.

3 Restriction and Frobenius reciprocity

First, we describe the transitivity properties of Res and Ind. If we have a sequence of subgroups $K \leq H \leq G$, then, as restriction of functions to subsets is transitive, we clearly have, for every G-representation V,

$$\boxed{\operatorname{Res}_K^G V = \operatorname{Res}_K^H \operatorname{Res}_H^G V}$$

The situation for Ind is similar but the argument is more complicated:

Proposition 3.1. *With $K \leq H \leq G$ as above, for every K-representation W,*

$$\boxed{\operatorname{Ind}_K^G W \cong \operatorname{Ind}_H^G \operatorname{Ind}_K^H W}$$

where the isomorphism is of G-representations.

Proof. By definition,

$$\operatorname{Ind}_K^G W = \{F\colon G \to W : F(gk) = \rho_W(k)^{-1}F(g)(\forall g \in G)(\forall k \in K)\};$$
$$\operatorname{Ind}_H^G \operatorname{Ind}_K^H W = \{F_1\colon G \to \operatorname{Ind}_K^H W : F(gh) = \rho_{\operatorname{Ind}_K^H}(h)^{-1}F_1(g)(\forall g \in G)(\forall h \in H)\}.$$

Thus in particular, if $F_1 \in \operatorname{Ind}_H^G \operatorname{Ind}_K^H W$, then $F_1(g)$ is itself a function from H to W, i.e. $F_1(g)(h) \in W$ and the function $F_1(g)$ must satisfy

$$F_1(g)(hk) = \rho_W(k)^{-1}F_1(g)(h) \qquad (\forall h \in H)(\forall k \in K)$$
$$F_1(gx)(h) = \rho_{\operatorname{Ind}_H^G}(x)^{-1}(F_1(g))(h) = F_1(g)(xh) \qquad (\forall g \in G)(\forall x, h \in H).$$

Define functions $A\colon \operatorname{Ind}_K^G W \to \operatorname{Ind}_H^G \operatorname{Ind}_K^H W$ and $B\colon \operatorname{Ind}_H^G \operatorname{Ind}_K^H W \to \operatorname{Ind}_K^G W$ by the formulas

$$A(F) = F_1, \qquad \text{where } F_1(g)(h) = F(gh)$$
$$B(F_1) = F, \qquad \text{where } F(g) = F_1(g)(1).$$

First, we claim that in fact $A(F) \in \operatorname{Ind}_H^G \operatorname{Ind}_K^H W$. In fact, for all $g \in G$, $h \in H$, $k \in K$, we have

$$A(F)(g)(hk) = F(ghk) = \rho_W(k)^{-1}F(gh) = \rho_W(k)^{-1}(F_1(g)(h)),$$

so that $A(F)(g) \in \operatorname{Ind}_K^H W$. Moreover, for all $x, h \in H$,

$$A(F)(gx)(h) = F(gxh) = A(F)(g)(xh),$$

in other words $A(F)(gx) = \rho_{\mathrm{Ind}_H^G}(x)^{-1}(A(F)(g))$. This says that $A(F) \in \mathrm{Ind}_H^G \mathrm{Ind}_K^H W$.

Similarly, if $F_1 \in \mathrm{Ind}_H^G \mathrm{Ind}_K^H W$, then for all $g \in G, h \in H, k \in K$,

$$B(F_1)(gh) = F_1(gh)(1) = \rho_{\mathrm{Ind}_H^G}(h)^{-1}(F_1(g))(1) = F_1(g)(h);$$
$$B(F_1)(gk) = F_1(g)(k) = \rho_W(k)^{-1}F_1(g)(1) = \rho_W(k)^{-1}B(F_1)(g).$$

The second line says $B(F_1) \in \mathrm{Ind}_K^G W$. Finally, using the first line above,

$$A(B(F_1))(g)(h) = B(F_1)(gh) = F_1(g)(h),$$

so that $A \circ B = \mathrm{Id}$. And

$$B(A(F))(g) = A(F)(g)(1) = F(g \cdot 1) = F(g),$$

so that $B \circ A = \mathrm{Id}$. Finally, we must check that one of A, B is a G-morphism, say A. But

$$A(\rho_{\mathrm{Ind}_K^G W}(x)F)(g)(h) = F(x^{-1}gh) = \rho_{\mathrm{Ind}_K^H W}(x)A(F)(g)(h).$$

Thus A is a G-morphism and a linear isomorphism, hence a G-isomorphism.

$\square$

We turn to various versions of Frobenius reciprocity:

Proposition 3.2 (Frobenius reciprocity I)**.** *Let H be a subgroup of G, let W be an H-representation, and let U be a G-representation. Then there is a linear isomorphism*

$$\boxed{\mathrm{Hom}^H(W, \mathrm{Res}_H^G U) \cong \mathrm{Hom}^G(\mathrm{Ind}_H^G W, U)}$$

In the language of category theory, this says that Res and Ind are adjoint functors.

A more quantitative version is the following:

Corollary 3.3 (Frobenius reciprocity II)**.** *Let H, G, W and U be as above. Denote by $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_G$ the Hermitian inner products on $L^2(H)$ and $L^2(G)$ respectively. Then*

$$\boxed{\langle \chi_W, \chi_{\mathrm{Res}_H^G U} \rangle_H = \langle \chi_{\mathrm{Ind}_H^G W}, \chi_U \rangle_G}$$

Proof. This follows from the previous version, since

$$\langle \chi_W, \chi_{\mathrm{Res}_H^G U} \rangle_H = \dim \mathrm{Hom}^H(W, \mathrm{Res}_H^G U)$$

and similarly for $\langle \chi_{\mathrm{Ind}_H^G W}, \chi_U \rangle_G$. $\square$

Corollary 3.4. *With notation as above, if f_1 is a class function on H and f_2 is a class function on G, then*

$$\langle f_1, \mathrm{Res}_H^G f_2 \rangle_H = \langle \mathrm{Ind}_H^G f_1, f_2 \rangle_G.$$

Proof. By the previous corollary, this holds for functions f_1 of the form χ_W and f_2 of the form χ_U. But the vector space of all class functions is spanned by the characters, so the formula holds for all class functions. $\square$

Example 3.5. (1) Let $H = \{1\}$ and let $W = \mathbb{C}$, the trivial representation. If U is an irreducible representation of G, with $\dim U = d$, say, then $\mathrm{Res}_{\{1\}}^G U$ is just $\mathbb{C}^d$, viewed as a representation of $\{1\}$, and $\langle \chi_{\mathbb{C}}, \chi_{\mathrm{Res}_{\{1\}}^G U} \rangle_{\{1\}} = 1 \cdot d = d$. On the other hand, we have seen that $\mathrm{Ind}_{\{1\}}^G \mathbb{C} = \mathbb{C}[G]$ is the regular representation, with character χ_{reg}. Moreover, since U is irreducible, $\langle \chi_{\mathrm{reg}}, \chi_U \rangle_G$ is the multiplicity of U in the regular representation. Thus we see again that this multiplicity is $\dim U$.

(2) Let U be an irreducible G-representation and let H be an arbitrary subgroup of G. Suppose that W is an irreducible summand of $\mathrm{Res}_H^G U$. Thus $\langle \chi_W, \chi_{\mathrm{Res}_H^G U} \rangle_H$ is the multiplicity of W in $\mathrm{Res}_H^G U$ and is a strictly positive integer. By Frobenius reciprocity, this multiplicity is equal to $\langle \chi_{\mathrm{Ind}_H^G W}, \chi_U \rangle_G$, which is the multiplicity of U in $\mathrm{Ind}_H^G W$. Thus U is in particular an irreducible summand of $\mathrm{Ind}_H^G W$.

As a special case, suppose that G is a nonabelian group of order pq, where p and q are primes and $p < q$. We have seen that necessarily $q \equiv 1 \bmod p$. Moreover, there is a normal subgroup H of G of order q. Finally, every irreducible representation of G has dimension 1 or p, and, up to isomorphism, there are p irreducible representations of dimension 1, identified with $\widehat{G/H}$, and $k = (q-1)/p$ irreducible representations of G of dimension p. Let U be an irreducible representation of G of dimension p. Then $\mathrm{Res}_H^G U$ is a representation of the abelian group H, and hence is a direct sum of 1-dimensional representations:

$$\mathrm{Res}_H^G U = \bigoplus_{i=1}^{p} W_i,$$

where $W_i = \mathbb{C}(\lambda_i)$ is a 1-dimensional representation corresponding to the homomorphism $\lambda_i\colon H \to \mathbb{C}^*$. As $\langle \chi_{W_i}, \chi_{\mathrm{Res}_H^G U}\rangle_H > 0$, it follows that U is isomorphic to a direct summand of $\mathrm{Ind}_H^G W_i$. But since $\dim U = p = \dim \mathrm{Ind}_H^G W_i$, we must have $U \cong \mathrm{Ind}_H^G W_i$. Hence every irreducible p-dimensional representation of G is of the form $\mathrm{Ind}_H^G W$ for a 1-dimensional representation W of H. The argument tells us a little more: since the multiplicity of U in $\mathrm{Ind}_H^G W_i$ is one, in the above notation, if $i \neq j$, then W_i is not isomorphic to W_j, i.e. $\lambda_i \neq \lambda_j$ for $i \neq j$. We leave it as a homework problem to give a more detailed analysis of the representations of G.

Proof of Frobenius reciprocity. As usual, we will define linear maps

$$\begin{aligned} A&\colon \mathrm{Hom}^H(W, \mathrm{Res}_H^G U) \to \mathrm{Hom}^G(\mathrm{Ind}_H^G W, U);\\ B&\colon \mathrm{Hom}^G(\mathrm{Ind}_H^G W, U) \to \mathrm{Hom}^H(W, \mathrm{Res}_H^G U)\end{aligned}$$

and verify that they are inverses. Given an element $f \in \mathrm{Hom}^H(W, \mathrm{Res}_H^G U)$, we have to define $A(f) \in \mathrm{Hom}^G(\mathrm{Ind}_H^G W, U)$. It suffices to define $A(f)$ on $F_{i,w}$ for every i, $1 \leq i \leq k$, and every $w \in W$, since every $F \in \mathrm{Ind}_H^G$ is uniquely written as $\sum_{i=1}^k F_{i,w}$. Define

$$A(f)(F_{i,w}) = \rho_U(x_i)f(w) \in U.$$

We claim that $A(f)$ is a G-morphism, hence that $A(f) \in \mathrm{Hom}^G(\mathrm{Ind}_H^G W, U)$. It suffices to check this on elements of the form $F_{i,w}$. Then, if $gx_i = x_j h_i(g)$,

$$\begin{aligned} A(f)(\rho_{\mathrm{Ind}_H^G W}(g)F_{i,w}) &= A(f)(F_{j,\rho_W(h_i(g))(w)}) = \rho_U(x_j)f(\rho_W(h_i(g))(w))\\ &= \rho_U(x_j)\rho_U(h_i(g))f(w) = \rho_U(x_j h_i(g))f(w)\\ &= \rho_U(gx_i)f(w) = \rho_U(g)\rho_U(x_i)f(w) = \rho_U(g)A(f)(F_{i,w}).\end{aligned}$$

This says that $A(f)$ is a G-morphism.

To define $B\colon \mathrm{Hom}^G(\mathrm{Ind}_H^G W, U) \to \mathrm{Hom}^H(W, \mathrm{Res}_H^G U)$, recall that we have $W^{(1)} \subseteq \mathrm{Ind}_H^G W$, and $W^{(1)}$ is an H-invariant subspace of $\mathrm{Ind}_H^G W$ which is H-isomorphic to W. Then, given $\Psi \in \mathrm{Hom}^G(\mathrm{Ind}_H^G W, U)$, define $B(\Psi) = \Psi|W^{(1)}\colon W^{(1)} \cong W \to U$. In other words, $B(\Psi)(w) = \Psi(F_{1,w})$. It satisfies: for all $h \in H$,

$$\begin{aligned} B(\Psi)(\rho_W(h)(w)) &= \Psi(F_{1,\rho_W(h)}) = \Psi(\rho_{\mathrm{Ind}_H^G W}(h)(F_{1,w})\\ &= \rho_U(h)\Psi(F_{1,w}) = \rho_{\mathrm{Res}_H^G U}(h)B(\Psi)(w).\end{aligned}$$

Thus $B(\Psi)$ is an H-morphism.

To show that $A \circ B = \mathrm{Id}$, by definition

$$\begin{aligned}(A \circ B)(\Psi)(F_{i,w}) &= \rho_U(x_i)B(\Psi)(w) = \rho_U(x_i)\Psi(F_{1,w}) \\ &= \Psi(\rho_{\mathrm{Ind}_H^G W}(x_i)F_{1,w}) = \Psi(F_{i,w}),\end{aligned}$$

since as previously noted $\rho_{\mathrm{Ind}_H^G W}(x_i)(F_{1,w}) = F_{i,w}$ (remark at the bottom of p. 5). Thus $A \circ B = \mathrm{Id}$. Finally, to see that $B \circ A = \mathrm{Id}$, note that

$$(B \circ A)(f)(w) = A(f)(F_{1,w}) = \rho_U(1)f(w) = f(w).$$

Thus $B \circ A = \mathrm{Id}$ and A, B are linear isomorphisms as claimed. $\square$

We remark that it is easy to give a direct proof of Frobenius reciprocity in the form of Corollary 3.4. We leave this as an exercise.

We conclude this section with a much easier formula:

Proposition 3.6 (Projection formula)**.** *If W is an H-representation and U is a G-representation, then, as G-representations,*

$$\boxed{\mathrm{Ind}_H^G(W \otimes \mathrm{Res}_H^G U) \cong \mathrm{Ind}_H^G W \otimes U}$$

Proof. The proposition implies (and is equivalent to) the corresponding formula for characters

$$\chi_{\mathrm{Ind}_H^G(W \otimes \mathrm{Res}_H^G U)} = \chi_{\mathrm{Ind}_H^G W} \cdot \chi_U.$$

More generally, this formula is implied by (and is equivalent to) the following: If f_1 is a class function on H and f_2 is a class function on G, then (in the notation of Definition 2.1)

$$\mathrm{Ind}_H^G(f_1 \mathrm{Res}_H^G f_2) = \mathrm{Ind}_H^G(f_1)f_2.$$

To prove this, first observe that (we write $\mathrm{Res}_H^G f_2 = f_2|H$ for brevity)

$$\widetilde{f_1(f_2|H)} = \tilde{f}_1 f_2.$$

Thus, for all $x, g \in G$,

$$\widetilde{f_1(f_2|H)}(x^{-1}gx) = \tilde{f}_1(x^{-1}gx)f_2(x^{-1}gx) = \tilde{f}_1(x^{-1}gx)f(g),$$

since f_2 is a class function on G. Then, for all $g \in G$,

$$\begin{aligned}\mathrm{Ind}_H^G(f_1 \mathrm{Res}_H^G f_2)(g) &= \frac{1}{\#(H)}\sum_{x\in G}\widetilde{f_1(f_2|H)}(x^{-1}gx) = \frac{1}{\#(H)}\sum_{x\in G}\tilde{f}_1(x^{-1}gx)f(g) \\ &= \left(\frac{1}{\#(H)}\sum_{x\in G}\tilde{f}_1(x^{-1}gx)\right)f(g) = \mathrm{Ind}_H^G(f_1)(g)f_2(g).\end{aligned}$$

This proves the formula. $\square$

Example 3.7. Suppose $W = \mathbb{C} = \mathbb{C}(1)$ is the trivial representation of H and V is a G-representation. Then $\mathbb{C} \otimes \operatorname{Res}_H^G V \cong \operatorname{Res}_H^G V$ and $\operatorname{Ind}_H^G \mathbb{C} \cong \mathbb{C}[G/H]$, the permutation representation associated to the action of G on G/H by left multiplication. Hence we get the following formula:

$$\boxed{\operatorname{Ind}_H^G \operatorname{Res}_H^G V \cong \mathbb{C}[G/H] \otimes V}$$

One can also ask for a corresponding formula when we compose Res and Ind in the opposite order. However, as we shall see, the formula for $\operatorname{Res}_H^G \operatorname{Ind}_H^G W$ is much more complicated.

4 Interpretation in terms of tensor products

Many of the arguments in the last section have much simpler proofs via general properties of tensor products of left and right modules over noncommutative rings. For example, Proposition 3.1 is just the isomorphism, for a $\mathbb{C}[K]$ module W,

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} (\mathbb{C}[H] \otimes_{\mathbb{C}[K]} W) \cong (\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}[H]) \otimes_{\mathbb{C}[K]} W \cong \mathbb{C}[G] \otimes_{\mathbb{C}[K]} W.$$

Frobenius reciprocity is just the natural isomorphism

$$\operatorname{Hom}^{\mathbb{C}[H]}(W, \operatorname{Res}_H^G U) \cong \operatorname{Hom}^{\mathbb{C}[G]}(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W, U).$$

Finally, the projection formula is the natural isomorphism

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} (W \otimes \operatorname{Res}_H^G U) \cong (\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W) \otimes U.$$