Permutation representations

1 Permutation representations

Let G be a finite group and let X be a finite G-set. For simplicity we will assume that $\#(X) \geq 2$. Recall that G acts *transitively* on X if, for all $x, y \in X$, there exists a $g \in G$ such that $g \cdot x = y$. Equivalently, there is exactly one G-orbit, i.e. for one (or equivalently all) $x \in X$, $G \cdot x = X$.

Definition 1.1. G acts *doubly transitively* on X if, for all $x, y, z, w \in X$ with $x \neq y$ and $z \neq w$, there exists a $g \in G$ such that $g \cdot x = z$ and $g \cdot y = w$. In particular, the G-action is transitive.

Equivalently, let G act on the Cartesian product $X \times X$ in the obvious way: $g \cdot (x, y) = (g \cdot x, g \cdot y)$, and let $\Delta \subseteq X \times X$ be the diagonal:

$$\Delta = \{(x, x) : x \in X\} = \{(x, y) \in X \times X : x = y\}.$$

Thus Δ is a G-invariant subset and so $X \times X - \Delta$ is also a G-set. Then G acts doubly transitively on $X \iff G$ acts transitively on $X \times X - \Delta$ $\iff$ there are exactly two G-orbits for the action of G on $X \times X$, namely $X \times X - \Delta$ and Δ.

Example 1.2. 1) The symmetric group S_n acts doubly transitively on $\{1, \ldots, n\}$ for $n \geq 2$. In fact, given $i, j, k, \ell \in \{1, \ldots, n\}$ with $i \neq j$ and $k \neq \ell$, the sets $\{1, \ldots, n\} - \{i, j\}$ and $\{1, \ldots, n\} - \{k, \ell\}$ both have $n - 2$ elements, so there is some bijection

$$\sigma_0 \colon \{1, \ldots, n\} - \{i, j\} \to \{1, \ldots, n\} - \{k, \ell\}.$$

Then define a permutation $\sigma \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ by

$$\sigma(i) = k; \qquad \sigma(j) = \ell; \qquad \sigma(x) = \sigma_0(x), \quad x \neq i, j.$$

Then by construction $\sigma(i) = k$ and $\sigma(j) = \ell$, so the action is doubly transitive.

2) The alternating group A_n acts doubly transitively on $\{1, \ldots, n\}$ for $n \geq 4$: Given $i, j, k, \ell \in \{1, \ldots, n\}$ with $i \neq j$ and $k \neq \ell$, use the above to find a $\sigma \in S_n$ such that $\sigma(i) = k$ and $\sigma(j) = \ell$. If $\sigma \in A_n$, we are done. Otherwise, σ is odd. Since $n \geq 4$, there exists two distinct elements $r, s \in \{1, \ldots, n\} - \{k, \ell\}$. Then $(r, s)\sigma \in A_n$ since it is a product of an even number of transpositions and $(r, s)\sigma(i) = (r, s)(k) = k$ and similarly $(r, s)\sigma(j) = (r, s)(\ell) = \ell$. Thus the action is doubly transitive. Note however that for $n = 2, 3$ the action of A_n on $\{1, \ldots, n\}$ is not doubly transitive. For example, there is no $\sigma \in A_3$ such that $\sigma(1) = 2$ and $\sigma(2) = 1$.

3) For $n \geq 4$, the action of D_n on the vertices of a regular n-gon (or equivalently on the n points $(\cos \frac{2\pi k}{n}, \sin \frac{2\pi k}{n})$ in the model for D_n we have constructed) is transitive but **not** doubly transitive. This is because the action must send a pair of adjacent vertices to a pair of adjacent vertices, and, for $n \geq 4$, there are always vertices which are not adjacent.

Let X be a G-set and consider $\mathbb{C}[X]$, viewed as a G-representation. Our goal is to better understand the decomposition of $\mathbb{C}[X]$ into irreducible representations. There is always a one-dimensional G-invariant subspace $W_1 = t \cdot \sum_{x \in X} x$ on which G acts trivially and a subspace

$$W_2 = \left\{ \sum_{x \in X} t_x \cdot x : \sum_{x \in X} t_x = 0 \right\},$$

with $\mathbb{C}[X] = W_1 \oplus W_2$. Note that $W_1 \subseteq \mathbb{C}[X]^G$, but equality does not always hold.

Proposition 1.3. *$\dim \mathbb{C}[X]^G$ is equal to the number of G-orbits of X.*

Proof. Write the distinct orbits of G in X as $O_1, \ldots, O_t$. For each i, $1 \leq i \leq t$, set

$$v_i = \sum_{x \in O_i} x \in \mathbb{C}[X].$$

Viewing $\mathbb{C}[X]$ as $L^2(X)$, the vector space of functions from X to $\mathbb{C}$, the element v_i corresponds to the *characteristic function of* O_i, i.e. the function f_{O_i} defined by

$$f_{O_i}(x) = \begin{cases} 1, & \text{if } x \in O_i; \\ 0, & \text{if } x \notin O_i. \end{cases}$$

Clearly $v_1, \ldots, v_t$ are linearly independent elements of $\mathbb{C}[X]$. Moreover,

$$\rho_{\mathbb{C}[X]}(g)(v_i) = \sum_{x \in O_i} \rho_{\mathbb{C}[X]}(g)(x) = \sum_{x \in O_i} g \cdot x = \sum_{x \in O_i} x = v_i,$$

since $g \in G$ permutes the orbit O_i. Hence $v_1, \dots, v_t$ are linearly independent elements of $\mathbb{C}[X]^G$. We must show that they span $\mathbb{C}[X]^G$. Given an element $\alpha = \sum_{x \in X} t_x \cdot x$, we can break the sum up into a sum over the orbits:

$$\alpha = \sum_{x \in X} t_x \cdot x = \sum_{i=1}^{t} \sum_{x \in O_i} t_x \cdot x.$$

Claim 1.4. *If $\alpha = \sum_{x \in X} t_x \cdot x \in \mathbb{C}[X]^G$, then, for all $x, y \in O_i$, $t_x = t_y$, i.e. the value t_x is the same for all $x \in O_i$.*

In fact, assuming the claim, let s_i be the common value of t_x for $x \in O_i$. Then $\alpha = \sum_{i=1}^{t} s_i v_i$. Thus the v_i span $\mathbb{C}[X]^G$ and hence are a basis, so $\dim \mathbb{C}[X]^G = t$. $\square$

Proof of the claim. It follows from the definitions that $\alpha = \sum_{x \in X} t_x \cdot x \in \mathbb{C}[X]^G \iff \rho_{\mathbb{C}[X]}(g)(\alpha) = \alpha$ for all $g \in G \iff$ for all $g \in G$

$$\sum_{x \in X} t_x \cdot (g \cdot x) = \sum_{x \in X} t_x \cdot x.$$

Equivalently, for all $x \in X$ and all $g \in G$, $t_x = t_{g^{-1} \cdot x}$. In particular, if y is in the same orbit O_i as x, say $x = g \cdot y$, then $t_y = t_{g^{-1} \cdot x} = t_x$, which is the statement of the claim. $\square$

Corollary 1.5. *The subspace W_1 of $\mathbb{C}[X]$ is equal to $\mathbb{C}[X]^G$, or equivalently $(W_2)^G = \{0\} \iff G$ acts transitively on X.* $\square$

Corollary 1.6 (Burnside's lemma)**.** *If the finite group G acts on a finite set X, and t is the number of G-orbits of X, then*

$$t\#(G) = \sum_{g \in G} \#(X^g).$$

Proof. By general theory,

$$\dim \mathbb{C}[X]^G = \langle \chi_{\mathbb{C}[X]}, 1 \rangle = \frac{1}{\#(G)} \sum_{g \in G} \chi_{\mathbb{C}[X]}(g) = \frac{1}{\#(G)} \sum_{g \in G} \#(X^g),$$

where $X^g = \{x \in X : g \cdot x = x\}$. Thus the corollary follows from Proposition 1.3. $\square$

Theorem 1.7. *With notation as above, write $\mathbb{C}[X] = W_1 \oplus W_2$. Then $W_1 = \mathbb{C}[X]^G \iff G$ acts transitively on X. Moreover, in this case W_2 is irreducible $\iff G$ acts doubly transitively on X.*

Proof. The first statement is Corollary 1.5. Assume that this is the case. Now W_2 is irreducible $\iff$ $\langle \chi_{W_2}, \chi_{W_2} \rangle = 1$. On the other hand, we can write

$$\chi_{\mathbb{C}[X]} = \chi_{W_1} + \chi_{W_2} = 1 + \chi_{W_2}.$$

Thus

$$\begin{aligned}\langle \chi_{\mathbb{C}[X]}, \chi_{\mathbb{C}[X]} \rangle &= \langle 1 + \chi_{W_2}, 1 + \chi_{W_2} \rangle \\ &= \langle 1, 1 \rangle + \langle 1, \chi_{W_2} \rangle + \langle \chi_{W_2}, 1 \rangle + \langle \chi_{W_2}, \chi_{W_2} \rangle.\end{aligned}$$

Clearly $\langle 1, 1 \rangle = 1$. Since G acts transitively on X, $W_2^G = \{0\}$, and hence $\langle \chi_{W_2}, 1 \rangle = 0$, likewise $\langle 1, \chi_{W_2} \rangle = \overline{\langle \chi_{W_2}, 1 \rangle} = 0$. It follows that

$$\langle \chi_{\mathbb{C}[X]}, \chi_{\mathbb{C}[X]} \rangle = 1 + \langle \chi_{W_2}, \chi_{W_2} \rangle.$$

Thus W_2 is irreducible $\iff$ $\langle \chi_{W_2}, \chi_{W_2} \rangle = 1$ $\iff$ $\langle \chi_{\mathbb{C}[X]}, \chi_{\mathbb{C}[X]} \rangle = 2$.

Clearly, all of the values of $\chi_{\mathbb{C}[X]}$ are integers, since $\chi_{\mathbb{C}[X]}(g) = \#(X^g)$. In particular, they are real numbers. Thus

$$\langle \chi_{\mathbb{C}[X]}, \chi_{\mathbb{C}[X]} \rangle = \frac{1}{\#(G)} \sum_{g \in G} |\chi_{\mathbb{C}[X]}(g)|^2 = \frac{1}{\#(G)} \sum_{g \in G} \chi_{\mathbb{C}[X]}(g)^2 = \langle \chi^2_{\mathbb{C}[X]}, 1 \rangle.$$

Next, we claim that

$$\chi^2_{\mathbb{C}[X]} = \chi_{\mathbb{C}[X \times X]}.$$

In fact, for every $g \in G$, $\chi^2_{\mathbb{C}[X]}(g) = (\#(X^g))^2$. On the other hand, we have seen that

$$\chi_{\mathbb{C}[X \times X]}(g) = \#((X \times X)^g),$$

where $(X \times X)^g = \{(x, y) \in X \times X : g \cdot (x, y) = (x, y)\}$. Since $g \cdot (x, y) = (g \cdot x, g \cdot y)$, $(x, y) \in (X \times X)^g \iff g \cdot x = x$ and $g \cdot y = y \iff x \in X^g$ and $y \in X^g$. In other words, $(X \times X)^g = (X^g) \times (X^g)$. Thus $\#((X \times X)^g) = (\#(X^g))^2$ and hence $\chi^2_{\mathbb{C}[X]} = \chi_{\mathbb{C}[X \times X]}$ as claimed.

Putting this together, we see that

$$\langle \chi_{\mathbb{C}[X]}, \chi_{\mathbb{C}[X]} \rangle = \langle \chi^2_{\mathbb{C}[X]}, 1 \rangle = \langle \chi_{\mathbb{C}[X \times X]}, 1 \rangle.$$

By Proposition 1.3, $\langle \chi_{\mathbb{C}[X \times X]}, 1 \rangle$ is the number of orbits of G acting on $X \times X$. Hence, by the remarks in Definition 1.1, $\langle \chi_{\mathbb{C}[X]}, \chi_{\mathbb{C}[X]} \rangle = 2 \iff$ the G-action on $X \times X$ has exactly two orbits $\iff$ the G-action on X is doubly transitive. $\square$

Corollary 1.8. *For $n \geq 2$, the representation of S_n on $W = \{(t_1, \dots, t_n) : \sum_i t_i = 0\}$ is an irreducible representation of dimension $n - 1$. For $n \geq 4$, the representation of A_n on W is an irreducible representation of dimension $n - 1$.* $\square$