Problem set 1 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Let V be a finite dimensional complex vector space. Let $T : V \to V$ be a finite order linear automorphism. Explain why T is diagonizable in your own words.

Exercise 2. Let V_1, V_2 be finite dimensional complex vector spaces. Let $T_i : V_i \to V_i$, $i = 1, 2$ be linear maps. Set $V = V_1 \oplus V_2$ (direct sum). Denote $T = T_1 \oplus T_2 : V \to V$ the induced linear map defined by the rule $T(v_1 \oplus v_2) = (T_1(v_1), T_2(v_2))$.

(1) Express the rank of T in terms of the ranks of T_1 and T_2.
(2) Express $\mathrm{Tr}(T)$ in terms of $\mathrm{Tr}(T_1)$ and $\mathrm{Tr}(T_2)$.
(3) Express $\det(T)$ in terms of $\det(T_1)$ and $\det(T_2)$.

Please explain briefly.

Exercise 3. Let V_1, V_2 be finite dimensional complex vector spaces. Let $T_i : V_i \to V_i$, $i = 1, 2$ be linear maps. Set $V = V_1 \otimes V_2$ (tensor product). Recall that elements v of V are finite linear combinations

$$v = \sum v_{1,k} \otimes v_{2,k}$$

subject to the relations

$$(v_1 + v_1') \otimes v_2 = v_1 \otimes v_2 + v_1' \otimes v_2, \quad v_1 \otimes (v_2 + v_2') = v_1 \otimes v_2 + v_1 \otimes v_2', \quad a v_1 \otimes v_2 = v_1 \otimes a v_2$$

Thus there is a unique linear map $T = T_1 \otimes T_2 : V \to V$ such that for all $v_i \in V_i$, $i = 1, 2$ we have $T(v_1 \otimes v_2) = T_1(v_1) \otimes T_2(v_2)$.

(1) Express the rank of T in terms of the ranks of T_1 and T_2.
(2) Express $\mathrm{Tr}(T)$ in terms of $\mathrm{Tr}(T_1)$ and $\mathrm{Tr}(T_2)$.
(3) Express $\det(T)$ in terms of $\det(T_1)$ and $\det(T_2)$.

Please explain briefly. Hint: If $e_1, \dots, e_n$ is a basis of V_1 and $f_1, \dots, f_m$ is a basis of V_2, then $e_i \otimes f_j$, $i = 1, \dots, n$, $j = 1, \dots, m$ is a basis for V. If it helps, first restrict to the case of diagonizable T_1 and T_2.

Exercise 4. Let $G = S_4$ be the symmetric group on 4 letters. Let $\rho : G \to GL_4(\mathbf{C})$ be the representation given by permutation matrices as in the first lecture.

(1) Compute the subspace of G-invariant vectors $V = (\mathbf{C}^4)^G$.
(2) As in the lecture, find a G-invariant subspace $W \subset \mathbf{C}^4$ such that $V \oplus W = \mathbf{C}^4$ (this is called an "internal direct sum" in the linear algebra notes).
(3) Show that W is irreducible.
(4) Discuss to what extent your arguments would work if 4 is replaced by $n > 4$.

Hints: two possible things you can try for (3): (a) you could show that for any nonzero vector x in W the span of the elements $\sigma \cdot x$ must be all of W, or (b) you could first show that no 1-dimensional subspace $W' \subset W$ can be G-invariant (as in the first lecture) and then show that if $W' \subset W$ is 2-dimensional, then it contains a nonzero vector x of the form $x = (x_1, x_2, x_3, 0)$ and use the result in 3 dimensions to conclude.

Exercise 5. For any integer $n > 1$ find a nonabelian group G of order $6n^3$ inside $GL_3(\mathbf{C})$. Is the representation you found irreducible?