Problem set 10 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Material from lecture on Thursday, November 16. Let me explain it a little bit more here.

Situation. Let $G$ be a finite group. For $i = 1, 2$, let $H_i \subset G$ be a subgroup and let $\chi_i : H_i \to \mathbb{C}^*$ be a group homomorphism (sometimes called a character, or a linear character, or a degree 1 character). Assume the following

(A1) $\chi_1|_{H_1 \cap H_2} = \chi_2|_{H_1 \cap H_2}$

(A2) for $g \in G$, $g \notin H_1H_2$ there exist a $z \in H_1 \cap gH_2g^{-1}$ such that $\chi_1(z) \neq \chi_2(g^{-1}zg)$.

Lemma 1. There exists a unique (up to isomorphism) irreducible representation $(V, \pi)$ of $G$ such that $(V, \pi)$ occurs in both $\text{Ind}_{H_1}^G \chi_1$ and $\text{Ind}_{H_2}^G \chi_2$.

Proof. It suffices to show that

$$\text{Hom}_G(\text{Ind}_{H_1}^G \chi_1, \text{Ind}_{H_2}^G \chi_2)$$

is 1-dimensional. To see this we use adjointness of functors

$$\text{Hom}_G(\text{Ind}_{H_1}^G \chi_1, \text{Ind}_{H_2}^G \chi_2) = \text{Hom}_{H_1}(\chi_1, \text{Res}_{H_1}^G \text{Ind}_{H_2}^G \chi_2)$$

Write

$$G = H_1g_1H_2 \cdots H_ng_nH_2$$

We may and do choose $g_1 = 1$. Then $g_i \notin H_1H_2$ for $i = 2, \ldots, n$. By Mackey’s second theorem we have

$$\text{Res}_{H_1}^G \text{Ind}_{H_2}^G \chi_2 = \text{Ind}_{H_1 \cap H_2}^G \chi_2|_{H_1 \cap H_2} \oplus \bigoplus_{i=2,\ldots,n} \text{Ind}_{H_1,i}^G \psi_i$$

where

$$H_{1,i} = H_1 \cap g_iH_2g_i^{-1} \quad \text{and} \quad \psi_i(z) = \chi_2(g_i^{-1}zg_i)$$

Thus we have

$$\text{Hom}_{H_1}(\chi_1, \text{Res}_{H_1}^G \text{Ind}_{H_2}^G \chi_2)$$

$$\quad = \text{Hom}_{H_1}(\chi_1, \text{Ind}_{H_1 \cap H_2}^G \chi_2|_{H_1 \cap H_2}) \oplus \bigoplus_{i=2,\ldots,n} \text{Hom}_{H_1,i}(\chi_1, \text{Ind}_{H_1,i}^G \psi_i)$$

$$\quad = \text{Hom}_{H_1 \cap H_2}(\chi_1|_{H_1 \cap H_2}, \chi_2|_{H_1 \cap H_2}) \oplus \bigoplus_{i=2,\ldots,n} \text{Hom}_{H_1,i}(\chi_1|_{H_1,i}, \psi_i)$$

where in the second equality we used adjointness of restriction and induction again.

Thus condition (A1) tells us that the first summand has dimension 1 and condition (A2) tells us that the other summand have dimension 0. QED

Lemma 2. The representation $(V, \pi)$ has multiplicity 1 in both $\text{Ind}_{H_1}^G \chi_1$ and $\text{Ind}_{H_2}^G \chi_2$.

Proof. Follows immediately from the fact that we proved $\text{Hom}_G(\text{Ind}_{H_1}^G \chi_1, \text{Ind}_{H_2}^G \chi_2)$ is 1-dimensional in the proof of Lemma 1. QED

Lemma 3. The representation $(V, \pi)$ is the only irreducible representation of $G$ (up to isomorphism) such that $\chi_1$ occurs in $\text{Res}_{H_1}^G (V, \pi)$ and $\chi_2$ occurs in $\text{Res}_{H_2}^G (V, \pi)$.

Proof. Follows from adjointness of restriction and induction. QED
For \( i = 1, 2 \) let us denote
\[
P_i = \sum_{h \in H_i} \chi_i^{-1}(h)\delta_h \in \mathbb{C}[G]
\]
Recall that an element \( T = \sum a_g \delta_g \in \mathbb{C}[G] \) acts on a representation \((W, \rho)\) of \( G \) by the rule \( W \ni w \mapsto T(w) = \sum a_g \rho(g)(w) \). Recall that \( T = 0 \Leftrightarrow T \) acts as the zero operator on each representation \( W \Leftrightarrow T \) acts as the zero operator on each irreducible representation \( W \). Recall that we have the convolution product \( * \) on \( \mathbb{C}[G] \) and that this product is compatible with the action of \( \mathbb{C}[G] \) on the representation \( W \).

**Lemma 4.** For any representation \((W, \rho)\) the subspace \( P_i(W) \) of \( W \) is nonzero if and only if \( \text{Hom}_{H_i}(\chi, \text{Res}^G_{H_i}W) \) is nonzero.

**Proof.** For \( h \in H_i \) we have
\[
\delta_h * P_i = \chi_i(h)P_i
\]
in \( \mathbb{C}[G] \). Hence if \( w = P_i(w') \) then \( \rho(h)(w) = \chi_i(h)w \) for \( h \in H_i \). Conversely, if \( w \in W \) satisfies \( \rho(h)(w) = \chi_i(h)w \), then \( P_i(w) = |H_i|w \) and we see that \( P_i \) is not the zero operator. **QED**

**Lemma 5.** For an irreducible representation \((W, \rho)\) of \( G \) the following are equivalent

1. \((W, \rho)\) is isomorphic to \((V, \pi)\),
2. \( P_1(W) \) and \( P_2(W) \) are nonzero,
3. \( P_1(P_2(W)) \) is nonzero.

**Proof.** The equivalence of (1) and (2) follows from Lemmas 1 and 4. If (3) holds, then (2) holds. In particular, we see that if \( P_1 * P_2 \) is nonzero, then it must act nontrivially on \( V \) and (2) and (3) are equivalent. To finish the proof write
\[
P_1 * P_2 = \sum_{h_1 \in H_1, h_2 \in H_2} \chi_1^{-1}(h_1)\chi_2^{-1}(h_2)\delta_{h_1 h_2}
\]
The coefficient of \( \delta_1 \) is
\[
\sum_{h \in H_1 \cap H_2} \chi_1^{-1}(h)\chi_2(h) = |H_1 \cap H_2|
\]
by assumption (A1) and hence \( P_1 * P_2 \) is nonzero.

**Lemma 6.** There exists a constant \( \mu \in \mathbb{C}^* \) such that
\[
\chi_{\pi}(g') = \mu \left( \sum_{g \in G, h_1 \in H_1, h_2 \in H_2, g' = gh_1 h_2^{-1}} \chi_1(h_1)\chi_2(h_2) \right)
\]
for all \( g' \in G \).

**Proof.** We claim that there exists a constant \( \mu \in \mathbb{C}^* \) such that
\[
\mu \left( \sum_{g \in G} \delta_g * P_1 * P_2 * \delta_{g^{-1}} \right) = \sum_{g \in G} \chi_{\pi}(g)\delta_g
\]
in \( \mathbb{C}[G] \). Of course, if this is true, then we see the lemma holds by looking at values of left and right hand side on \( g' \in G \). The displayed equality follows from the following three observations

1. Both right and left hand side of the equation are in the center of \( \mathbb{C}[G] \), i.e., they are class functions on \( G \).
2. Both right and left hand side act as 0 on each irreducible representation of \( G \), except on \( V \).
3. The expression \( \sum_{g \in G} \delta_g * P_1 * P_2 * \delta_{g^{-1}} \) is nonzero.
Namely, assume (1), (2), and (3) hold. By (1) both $T = \sum_{g \in G} \delta_g \ast P_1 \ast P_2 \ast \delta_{g^{-1}}$ and $T' = \sum_{g \in G} \chi_\pi(g) \delta_g$ act as a scalar on each irreducible representation $W$ of $G$. By (2) this scalar is zero, except for $W = V$. By (3) there exists an $\mu$ such that $\mu T$ and $T'$ act by the same scalar on $V$ as well. Then $T' - \mu T$ acts as zero on each irreducible representation of $G$ and hence $T' - \mu T$ is zero in $C[G]$.

Proof of (1), (2), and (3). Part (1) is left to the reader. Part (2) holds for the left hand side by Lemma 5 and for the right hand side because $\sum_{g \in G} \chi_\pi(g) \delta_g$ is up to a scalar the projection onto the $V$-isotypical component (see lectures). Part (3) holds by the computation in the proof of Lemma 5. QED

**Lemma 7.** For $g \in G$ denote $C_g$ the conjugacy class of $g$. We have

$$\chi_\pi(g) = \frac{\dim(V)}{|C_g| \cdot |H_1 \cap H_2|} \left( \sum_{h_1 \in H_1, h_2 \in H_2, h_1 h_2 \in C_g} \chi_1(h_1) \chi_2(h_2) \right)$$

This agrees with Burrow’s article “A generalization of the Young diagram”.

**Proof.** A counting argument using the result of Lemma 6 gives that there exists a constant $\mu' \in C^*$ such that for all $g \in G$ we have

$$\chi_\pi(g) = \frac{\mu'}{|C_g|} \left( \sum_{h_1 \in H_1, h_2 \in H_2, h_1 h_2 \in C_g} \chi_1(h_1) \chi_2(h_2) \right)$$

where $C_g \subset G$ is the conjugacy class of $g$. Evaluating this for $g = 1$ using (A1) we conclude that

$$\dim(V) = \mu' |H_1 \cap H_2|$$

Filling this into the formula above we get the lemma. QED

**Application to symmetric groups.** Let $G = S_n$. Let $\lambda \vdash n$ be a partition of $n$. Let $t_0$ be the basic tableau of type $\lambda$. We want to consider the case

$$H_1 = S_{\lambda} = R = R_{t_0} = \text{row stabilizer of } t_0$$

with $\chi_1 = 1$ the trivial character and

$$H_2 = C = C_{t_0} = \text{column stabilizer of } t_0$$

with $\chi_2 = \epsilon$ the sign character.

**Result.** $H_1 \cap H_2 = \{1\}$. This we discussed in class and everyone agreed. This proves that (A1) holds.

**Example.** Let $G = S_3$ and let $\lambda = (2,1)$. Then we have $H_1 = R = \{1, (12)\}$ and $H_2 = C = \{1, (13)\}$. The products $h_1 \cdot h_2$ for $h_1 \in H_1$ and $h_2 \in H_2$ are

$$1 \cdot 1 = 1, \quad (12) \cdot 1 = (12), \quad 1 \cdot (13) = (13), \quad (12) \cdot (13) = (132)$$

The elements $\sigma \not\in H_1 H_2 = RC$ are the elements

$$\sigma = (23), (123)$$

We have $H_1 \cap \sigma H_2 \sigma^{-1} = H_1$ in both cases. Thus condition (A2) holds. Let $(V, \pi)$ the the corresponding irreducible representation of $G = S_3$. Then we see that

$$\chi_\pi(1) = \dim(V), \quad \chi_\pi((12)) = 0, \quad \chi_\pi((123)) = -\frac{\dim(V)}{2}$$

The zero in the middle comes from the fact that we are summing the values of the sign character on the $h_2$ for those pairs $(h_1, h_2) \in H_1 \times H_2$ such that $h_1 \cdot h_2$ is in the
conjunctly class of (12). To get \( \text{dim}(V) \) we use that \((\chi_\pi, \chi_\pi) = 1\) for our irreducible character \( \chi_\pi \) which in this case means that

\[
1 = \frac{1}{6} \text{dim}(V)^2 \left(1 + 0 + \frac{1}{4}\right) = \frac{\text{dim}(V)^2}{4}
\]

We conclude that \( \text{dim}(V) = 2 \) which gives the usual character of the usual 2-dimensional irreducible representation.

**Notation.** For any tableau \( t \) of type \( \lambda \) we set

\[
R_t = \text{row stabilizer of } t \quad \text{and} \quad C_t = \text{column stabilizer of } t
\]

Note that given a tableau \( t \) of type \( \lambda \) there is a unique \( \sigma \in G \) such that \( t = \sigma(t_0) \) and then we have

\[
C_t = \sigma C \sigma^{-1} = \sigma H_2 \sigma^{-1}
\]

**Exercise 1.** Prove that (A2) is equivalent to

(S1) For every \( g \in G, \ g \notin H_1 H_2 \) the group \( H_1 \cap gH_2 g^{-1} \) contains an odd permutation.

**Exercise 2.** Prove that (S1) is equivalent to

(S1') For every \( t = g(t_0) \) with \( g \in G, \ g \notin RC \) the group \( R \cap C_t \) contains an odd permutation.

**Exercise 3.** Prove that (S1') is equivalent to (S2) + (S3) which are as follows

(S2) For every tableau \( t \) of type \( \lambda \), if \( R \cap C_t \neq \{1\} \), then \( R \cap C_t \) contains an odd permutation.

(S3) For every tableau \( t \) of type \( \lambda \), if \( R \cap C_t = \{1\} \), then \( t = \sigma(t_0) \) with \( \sigma \in RC \).

**Exercise 4.** Prove

(S2') For every tableau \( t \) of type \( \lambda \), if \( R \cap C_t \neq \{1\} \), then \( R \cap C_t \) contains a transposition.

which of course implies (S2).

**Exercise 5.** Prove that (S3) follows from

(S3') For every tableau \( t \) of type \( \lambda \) with increasing numbers down the columns, if \( R \cap C_t = \{1\} \), then \( t = r(t_0) \) with \( r \in R \).

**Answer to Exercise 5.** Assume (S3') holds. Let \( t \) be an arbitrary tableau with \( R \cap C_t = \{1\} \). Write \( t = \sigma(t_0) \). Let \( t' \) be the tableau which is column equivalent to \( t \) and with increasing numbers down the columns. Note that \( C_{t'} = C_t \) and hence we have \( R \cap C_{t'} = \{1\} \). By assumption, there exists an \( r \in R \) with \( t' = r(t_0) \). Write \( t' = c'(t) \) with \( c' \in C_t \). Since \( C_t = \sigma C \sigma^{-1} \), we see that \( c' = \sigma c \sigma^{-1} \) for some \( c \in C \). Then we see that

\[
\sigma(t_0) = t = (c')^{-1}(t') = \sigma(c^{-1}(\sigma^{-1}(r(t_0))))
\]

and hence we see that

\[
\sigma = \sigma c^{-1} \sigma^{-1} r \Rightarrow 1 = c^{-1} \sigma^{-1} r \Rightarrow cr^{-1} = \sigma^{-1} \Rightarrow \sigma = rc^{-1}
\]

which tells us that \( \sigma \in RC \) as desired. QED

**Exercise 6.** Prove (S3').

**Answer to Exercise 6.** We will try this in class on Tuesday, November 21.