Problem set 10 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Material from lecture on Thursday, November 16. Let me explain it a little bit more here.

Situation. Let G be a finite group. For $i = 1, 2$, let $H_i \subset G$ be a subgroup and let $\chi_i : H_i \to \mathbf{C}^*$ be a group homomorphism (sometimes called a character, or a linear character, or a degree 1 character). Assume the following

(A1) $\chi_1|_{H_1 \cap H_2} = \chi_2|_{H_1 \cap H_2}$
(A2) for $g \in G$, $g \notin H_1 H_2$ there exist a $z \in H_1 \cap gH_2g^{-1}$ such that $\chi_1(z) \neq \chi_2(g^{-1}zg)$.

Lemma 1. There exists a unique (up to isomorphism) irreducible representation (V, π) of G such that (V, π) occurs in both $\mathrm{Ind}_{H_1}^G \chi_1$ and $\mathrm{Ind}_{H_2}^G \chi_2$.

Proof. It suffices to show that

$$\mathrm{Hom}_G(\mathrm{Ind}_{H_1}^G \chi_1, \mathrm{Ind}_{H_2}^G \chi_2)$$

is 1-dimensional. To see this we use adjointness of functors

$$\mathrm{Hom}_G(\mathrm{Ind}_{H_1}^G \chi_1, \mathrm{Ind}_{H_1}^G \chi_1) = \mathrm{Hom}_{H_1}(\chi_1, \mathrm{Res}_{H_1}^G \mathrm{Ind}_{H_2}^G \chi_2)$$

Write

$$G = H_1 g_1 H_2 \amalg \ldots \amalg H_1 g_n H_2$$

We may and do choose $g_1 = 1$. Then $g_i \notin H_1 H_2$ for $i = 2, \ldots, n$. By Mackey's second theorem we have

$$\mathrm{Res}_{H_1}^G \mathrm{Ind}_{H_2}^G \chi_2 = \mathrm{Ind}_{H_1 \cap H_2}^{H_1} \chi_2|_{H_1 \cap H_2} \oplus \bigoplus\nolimits_{i=2,\ldots,n} \mathrm{Ind}_{H_{1,i}}^{H_1} \psi_i$$

where

$$H_{1,i} = H_1 \cap g_i H_2 g_i^{-1} \quad \text{and} \quad \psi_i(z) = \chi_2(g_i^{-1} z g_i)$$

Thus we have

$$\begin{aligned} &\mathrm{Hom}_{H_1}(\chi_1, \mathrm{Res}_{H_1}^G \mathrm{Ind}_{H_2}^G \chi_2) \\ &= \mathrm{Hom}_{H_1}(\chi_1, \mathrm{Ind}_{H_1 \cap H_2}^{H_1} \chi_2|_{H_1 \cap H_2}) \oplus \bigoplus\nolimits_{i=2,\ldots,n} \mathrm{Hom}_{H_1}(\chi_1, \mathrm{Ind}_{H_{1,i}}^{H_1} \psi_i) \\ &= \mathrm{Hom}_{H_1 \cap H_2}(\chi_1|_{H_1 \cap H_2}, \chi_2|_{H_1 \cap H_2}) \oplus \bigoplus\nolimits_{i=2,\ldots,n} \mathrm{Hom}_{H_{1,i}}(\chi_1|_{H_{1,i}}, \psi_i) \end{aligned}$$

where in the second equality we used adjointness of restriction and induction again. Thus condition (A1) tells us that the first summand has dimension 1 and condition (A2) tells us that the other summand have dimension 0. **QED**

Lemma 2. The representation (V, π) has multiplicity 1 in both $\mathrm{Ind}_{H_1}^G(\chi_1)$ and $\mathrm{Ind}_{H_2}^G(\chi_2)$.

Proof. Follows immediately from the fact that we proved $\mathrm{Hom}_G(\mathrm{Ind}_{H_1}^G \chi_1, \mathrm{Ind}_{H_2}^G \chi_2)$ is 1-dimensional in the proof of Lemma 1. **QED**

Lemma 3. The representation (V, π) is the only irreducible representation of G (up to isomorphism) such that χ_1 occurs in $\mathrm{Res}_{H_1}^G(V, \pi)$ and χ_2 occurs in $\mathrm{Res}_{H_2}^G(V, \pi)$.

Proof. Follows from adjointness of restriction and induction. **QED**

For $i = 1, 2$ let us denote

$$P_i = \sum\nolimits_{h \in H_i} \chi_i^{-1}(h)\delta_h \in \mathbf{C}[G]$$

Recall that an element $T = \sum a_g \delta_g \in \mathbf{C}[G]$ acts on a representation (W, ρ) of G by the rule $W \ni w \mapsto T(w) = \sum a_g \rho(g)(w)$. Recall that $T = 0 \Leftrightarrow T$ acts as the zero operator on each representation $W \Leftrightarrow T$ acts as the zero operator on each irreducible representation W. Recall that we have the convolution product $\star$ on $\mathbf{C}[G]$ and that this product is compatible with the action of $\mathbf{C}[G]$ on the representation W.

Lemma 4. For any representation (W, ρ) the subspace $P_i(W)$ of W is nonzero if and only if $\mathrm{Hom}_{H_i}(\chi_i, \mathrm{Res}^G_{H_i} W)$ is nonzero.

Proof. For $h \in H_i$ we have

$$\delta_h \star P_i = \chi_i(h) P_i$$

in $\mathbf{C}[G]$. Hence if $w = P_i(w')$ then $\rho(h)(w) = \chi_i(h)w$ for $h \in H_i$. Converesely, if $w \in W$ satisfies $\rho(h)(w) = \chi_i(h)w$, then $P_i(w) = |H_i|w$ and we see that P_i is not the zero operator. **QED**

Lemma 5. For an irreducible representation (W, ρ) of G the following are equivalent

(1) (W, ρ) is isomorphic to (V, π),
(2) $P_1(W)$ and $P_2(W)$ are nonzero,
(3) $P_1(P_2(W))$ is nonzero.

Proof. The equivalence of (1) and (2) follows from Lemmas 1 and 4. If (3) holds, then (2) holds. In particular, we see that if $P_1 \star P_2$ is nonzero, then it must act nontrivially on V and (2) and (3) are equivalent. To finish the proof write

$$P_1 \star P_2 = \sum\nolimits_{h_1 \in H_1,\ h_2 \in H_2} \chi_1^{-1}(h_1)\chi_2^{-1}(h_2)\delta_{h_1 h_2}$$

The coefficient of δ_1 is

$$\sum\nolimits_{h \in H_1 \cap H_2} \chi_1^{-1}(h)\chi_2(h) = |H_1 \cap H_2|$$

by assumption (A1) and hence $P_1 \star P_2$ is nonzero.

Lemma 6. There exists a constant $\mu \in \mathbf{C}^*$ such that

$$\chi_\pi(g') = \mu \left(\sum\nolimits_{g \in G,\ h_1 \in H_1,\ h_2 \in H_2,\ g' = g h_1 h_2 g^{-1}} \chi_1(h_1)\chi_2(h_2) \right)$$

for all $g' \in G$.

Proof. We claim that there exists a constant $\mu \in \mathbf{C}^*$ such that

$$\mu \left(\sum\nolimits_{g \in G} \delta_g \star P_1 \star P_2 \star \delta_{g^{-1}} \right) = \sum\nolimits_{g \in G} \overline{\chi_\pi(g)} \delta_g$$

in $\mathbf{C}[G]$. Of course, if this is true, then we see the lemma holds by looking at values of left and right hand side on $g' \in G$. The displayed equality follows from the following three observations

(1) Both right and left hand side of the equation are in the center of $\mathbf{C}[G]$, i.e., they are class functions on G.
(2) Both right and left hand side act as 0 on each irreducible representation of G, except on V.
(3) The expression $\sum_{g \in G} \delta_g \star P_1 \star P_2 \star \delta_{g^{-1}}$ is nonzero.

Namely, assume (1), (2), and (3) hold. By (1) both $T = \sum_{g\in G} \delta_g \star P_1 \star P_2 \star \delta_{g^{-1}}$ and $T' = \sum_{g\in G} \overline{\chi_\pi(g)}\delta_g$ act as a scalar on each irreducible representation W of G. By (2) this scalar is zero, except for $W = V$. By (3) there exists an μ such that μT and T' act by the same scalar on V as well. Then $T' - \mu T$ acts as zero on each irreducible representation of G and hence $T' - \mu T$ is zero in $\mathbf{C}[G]$.

Proof of (1), (2), and (3). Part (1) is left to the reader. Part (2) holds for the left hand side by Lemma 5 and for the right hand side because $\sum_{g\in G} \overline{\chi_\pi(g)}\delta_g$ is up to a scalar the projection onto the V-isotypical component (see lectures). Part (3) holds by the computation in the proof of Lemma 5. **QED**

Lemma 7. For $g \in G$ denote C_g the conjugacy class of g. We have

$$\chi_\pi(g) = \frac{\dim(V)}{|C_g| \cdot |H_1 \cap H_2|} \left(\sum\nolimits_{h_1\in H_1,\ h_2\in H_2,\ h_1h_2\in C_g} \chi_1(h_1)\chi_2(h_2) \right)$$

This agrees with Burrow's article "A generalization of the Young diagram".

Proof. A counting argument using the result of Lemma 6 gives that there exists a constant $\mu' \in \mathbf{C}^*$ such that for all $g \in G$ we have

$$\chi_\pi(g) = \frac{\mu'}{|C_g|} \left(\sum\nolimits_{h_1\in H_1,\ h_2\in H_2,\ h_1h_2\in C_g} \chi_1(h_1)\chi_2(h_2) \right)$$

where $C_g \subset G$ is the conjugacy class of g. Evaluating this for $g = 1$ using (A1) we conclude that

$$\dim(V) = \mu'|H_1 \cap H_2|$$

Filling this into the formula above we get the lemma. **QED**

Application to symmetric groups. Let $G = S_n$. Let $\lambda \vdash n$ be a partition of n. Let t_0 be the basic tableau of type λ. We want to consider the case

$$H_1 = S_\lambda = R = R_{t_0} = \text{row stabilizer of } t_0$$

with $\chi_1 = 1$ the trivial character and

$$H_2 = C = C_{t_0} = \text{column stabilizer of } t_0$$

with $\chi_2 = \epsilon$ the sign character.

Result. $H_1 \cap H_2 = \{1\}$. This we discussed in class and everyone agreed. This proves that (A1) holds.

Example. Let $G = S_3$ and let $\lambda = (2, 1)$. Then we have $H_1 = R = \{1, (12)\}$ and $H_2 = C = \{1, (13)\}$. The products $h_1 \cdot h_2$ for $h_1 \in H_1$ and $h_2 \in H_2$ are

$$1 \cdot 1 = 1, \quad (12) \cdot 1 = (12), \quad 1 \cdot (13) = (13), \quad (12) \cdot (13) = (132)$$

The elements $\sigma \notin H_1H_2 = RC$ are the elements

$$\sigma = (23),\ (123)$$

We have $H_1 \cap \sigma H_2 \sigma^{-1} = H_1$ in both cases. Thus condition (A2) holds. Let (V, π) the the corresponding irreducible representation of $G = S_3$. Then we see that

$$\chi_\pi(1) = \dim(V), \quad \chi_\pi((12)) = 0, \quad \chi_\pi((123)) = -\frac{\dim(V)}{2}$$

The zero in the middle comes from the fact that we are summing the values of the sign character on the h_2 for those pairs $(h_1, h_2) \in H_1 \times H_2$ such that $h_1 \cdot h_2$ is in the conjugacy class of (12). To get $\dim(V)$ we use that $(\chi_\pi, \chi_\pi) = 1$ for our irreducible character χ_π which in this case means that

$$1 = \frac{1}{6}\dim(V)^2\left(1 + 0 + 2\frac{1}{4}\right) = \frac{\dim(V)^2}{4}$$

We conclude that $\dim(V) = 2$ which gives the usual character of the usual 2-dimensional irreducible representation.

Notation. For any tableau t of type λ we set

$$R_t = \text{row stabilizer of } t \quad \text{and} \quad C_t = \text{column stabilizer of } t$$

Note that given a tableau t of type λ there is a unique $\sigma \in G$ such that $t = \sigma(t_0)$ and then we have

$$C_t = \sigma C \sigma^{-1} = \sigma H_2 \sigma^{-1}$$

Exercise 1. Prove that (A2) is equivalent to

(S1) For every $g \in G$, $g \notin H_1 H_2$ the group $H_1 \cap g H_2 g^{-1}$ contains an odd permutation.

Exercise 2. Prove that (S1) is equivalent to

(S1') For every $t = g(t_0)$ with $g \in G$, $g \notin RC$ the group $R \cap C_t$ contains an odd permutation.

Exercise 3. Prove that (S1') is equivalent to (S2) + (S3) which are as follows

(S2) For every tableau t of type λ, if $R \cap C_t \neq \{1\}$, then $R \cap C_t$ contains an odd permutation.

(S3) For every tableau t of type λ, if $R \cap C_t = \{1\}$, then $t = \sigma(t_0)$ with $\sigma \in RC$.

Exercise 4. Prove

(S2') For every tableau t of type λ, if $R \cap C_t \neq \{1\}$, then $R \cap C_t$ contains a transposition.

which of course implies (S2).

Exercise 5. Prove that (S3) follows from

(S3') For every tableau t of type λ with increasing numbers down the columns, if $R \cap C_t = \{1\}$, then $t = r(t_0)$ with $r \in R$.

Answer to Exercise 5. Assume (S3') holds. Let t be an arbitrary tableau with $R \cap C_t = \{1\}$. Write $t = \sigma(t_0)$. Let t' be the tableau which is column equivalent to t and with increasing numbers down the columns. Note that $C_{t'} = C_t$ and hence we have $R \cap C_{t'} = \{1\}$. By assumption, there exists an $r \in R$ with $t' = r(t_0)$. Write $t' = c'(t)$ with $c' \in C_t$. Since $C_t = \sigma C \sigma^{-1}$, we see that $c' = \sigma c \sigma^{-1}$ for some $c \in C$. Then we see that

$$\sigma(t_0) = t = (c')^{-1}(t') = \sigma(c^{-1}(\sigma^{-1}(r(t_0))))$$

and hence we see that

$$\sigma = \sigma c^{-1} \sigma^{-1} r \Rightarrow 1 = c^{-1}\sigma^{-1} r \Rightarrow c r^{-1} = \sigma^{-1} \Rightarrow \sigma = r c^{-1}$$

which tells us that $\sigma \in RC$ as desired. **QED**

Exercise 6. Prove (S3').

Answer to Exercise 6. We will try this in class on Tuesday, November 21.