Problem set 2 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

**Exercise 1.** Let \( k \) be a finite field with \( q \) elements. If you like, you may take \( k = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \) where \( p \) is a prime number and take \( q = p \). Let \( V \) be a vector space of dimension \( n \geq 1 \) over \( k \).

1. Explain why \( GL(V) \) is a finite group.
2. Compute the order of \( GL(V) \) in terms of \( n \) and \( q \).

Denote \( SL(V) \) the subgroup \( GL(V) \) consisting of elements whose determinant is 1.

3. Explain why \( SL(V) \) is a normal subgroup of \( GL(V) \).
4. Describe the group \( GL(V)/SL(V) \).
5. How many elements does \( SL(V) \) have?

**Exercise 2.** Let \( G \) be a finite group. In each of the following cases, explain briefly why there does not exist a finite dimensional representation \( \pi \) of \( G \) with character \( \chi_\pi \) having the stated properties:

1. \( \chi_\pi(1) = -1 \) where 1 \( \in \) \( G \) is the identity element.
2. \( \chi_\pi(1) = 1/2 \).
3. \( \chi_\pi(1) = 5 \) and \( \chi_\pi(g) = 6 \) for some \( g \in G \).
4. \( \chi_\pi(1) = 2 \) and \( \chi_\pi(g) = 1/11 \) for some \( g \in G \).
5. \( \chi_\pi(g) = 4 \) and \( \chi_\pi(g^{-1}) = -4 \) for some \( g \in G \).

**Exercise 3.** Let \( G \) be a finite group. Let \( X \) be a finite set. Let \( G \times X \to X \), \((g, x) \mapsto g \cdot x \) be an action of \( G \) on \( X \). Let \( C[X] \) be the corresponding permutation representation of \( G \). What this means is this:

(a) as a vector space \( C[X] = \{ \text{maps } f : X \to C \} \)
(b) for \( f \in C[X] \) and \( g \in G \) we define \( g \cdot f \) by the rule

\[
(g \cdot f)(x) = f(g^{-1} \cdot x)
\]
for all \( x \in X \).

Carefully explain why

1. the inverse in the formula is necessary,
2. the delta functions \( \delta_x \in C[X] \) where \( x \in X \) form a basis for \( C[X] \), and
3. \( g \cdot \delta_x = \delta_{g \cdot x} \).

**Remark.** Often people think of elements of \( C[X] \) as formal linear sums \( \xi = \sum t_x \cdot x \) with \( t_x \in C \). In other words, they think of \( C[X] \) as a \( C \)-vector space with basis given by the elements of \( X \). Then they define the \( G \)-action by the rule \( g \cdot \xi = \sum t_x g \cdot x \). This version is isomorphic to ours in the exercise above, via the maps sending the element \( \xi = \sum t_x \cdot x \) to the function \( f = \sum t_x \delta_x \).

**Exercise 4.** Let us call a representation isomorphic to one of the representations of Exercise 3 a permutation representation.

1. Give an example of a finite group \( G \) and a (finite dimensional as always) representation \( V \) which is not a permutation representation.
2. Show that if \( V_1 \) and \( V_2 \) are permutation representations of the same finite group \( G \), then so is \( V_1 \oplus V_2 \).
3. Show that if \( V_1 \) and \( V_2 \) are permutation representations of the same finite group \( G \), then so is \( V_1 \otimes V_2 \).
(4) Give an example of a group $G$ and a permutation representation $V$ such that $\wedge^2(V)$ is not a permutation representation. (If you solve this, then you’ve solved part (1) as well.)

(5) Show that a permutation representation is isomorphic to its dual. Hint: you may use that a representation $V$ is isomorphic to its dual if and only if there exists a $G$-invariant nondegenerate bilinear pairing $\langle , \rangle : V \times V \to \mathbb{C}$.

Exercise 5. Let $n \geq 1$. Let $\zeta = \exp(2i\pi/n)$ be the usual primitive $n$th root of 1. Consider the $n \times n$ matrix

$$A = \text{diag}(1, \zeta, \zeta^2, \ldots) = \begin{pmatrix}
1 & 0 & 0 & \ldots \\
0 & \zeta & 0 & \ldots \\
0 & 0 & \zeta^2 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

and the permutation matrix corresponding to the $n$-cycle $(12\ldots n)$, namely

$$B = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & \ldots \\
0 & 1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

Prove that

(1) $A$ and $B$ generate a finite subgroup $G$ of $GL_n(\mathbb{C})$

(2) the representation of $G$ on $\mathbb{C}^n$ you get in this way is irreducible.