Problem set 5 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Consider the subgroup $H = S_5 \subset S_6 = G$. Let $\chi : S_5 \to \{\pm 1\} \subset \mathbf{C}^*$ be the sign character which we also may and do view as a 1-dimensional irreducible representation of H. Let $\pi : S_5 \to GL_4(\mathbf{C})$ be the usual 4-dimensional irreducible representation (which is a summand of the standard permutation representation).

(1) Compute the character of the representation $\mathrm{Ind}_H^G(\chi)$.
(2) Compute the character of the representation $\mathrm{Ind}_H^G(\pi)$.
(3) How many irreducible constituents does $\mathrm{Ind}_H^G(\chi)$ have?
(4) How many irreducible constituents does $\mathrm{Ind}_H^G(\pi)$ have?

Exercise 2. Let H be a subgroup of a finite group G. Let V, W be nonzero representations of H. Using the universal property, show that there is a nonzero map

$$\mathrm{Ind}_H^G(V \otimes W) \longrightarrow \mathrm{Ind}_H^G(V) \otimes \mathrm{Ind}_H^G(W)$$

of representations of G.

Exercise 3. Fix an integer $n \geq 2$. Given a complex $n \times n$ matrix $A = (a_{ij})$ we define

$$|A| = \max |a_{ij}|$$

Below, we denote 1 the $n \times n$ identity matrix and A and B are complex $n \times n$ matrices. You can skip parts (1) – (5) if they are obvious to you.

(1) Briefly explain why $|A + B| \leq |A| + |B|$.
(2) Briefly explain why $|A + B| \geq |A| - |B|$.
(3) Briefly explain why $|AB| \leq n|A||B|$.
(4) Show that $|1 - BA| \leq |1 - A| + |1 - B| + n|1 - A||1 - B|$.
(5) Show that $|AB| \geq (1 - n|1 - A|)|B|$.
(6) Prove that for A and B invertible and $|1 - A| \leq \epsilon_1$ and $|1 - B| \leq \epsilon_2$ we have

$$|1 - ABA^{-1}B^{-1}| \leq f(\epsilon_1, \epsilon_2) = \frac{2n\epsilon_1\epsilon_2}{1 - n\epsilon_1 - n\epsilon_2 - n^2\epsilon_1\epsilon_2}$$

Hint: set $U = 1 - ABA^{-1}B^{-1}$ and $V = BA - AB = (1 - B)(1 - A) - (1 - A)(1 - B)$. Then $V = UBA$ and hence by (4) and (5) we can find a lower bound for $|V|$ in terms of $|U|$ and ϵ_1, ϵ_2. But we also have an upper bound for $|V|$ by (1) and (3).

Next, let $G \subset GL_n(\mathbf{C})$ be a finite subgroup. For an $\epsilon \geq 0$ define

$$H_\epsilon = \text{subgroup of } G \text{ generated by } g \in G \text{ such that } |1 - g| \leq \epsilon$$

Because G is finite, we obtain a sequence

$$0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_r < \epsilon_{r+1} = \infty$$

of real numbers where the groups H_ϵ jump. In other words, we have a sequence of subgroups

$$\{1\} = H_0 \subsetneq H_1 \subsetneq H_2 \subsetneq \ldots \subsetneq H_r = G$$

and we have $H_\epsilon = H_i$ if and only if $\epsilon \in [\epsilon_i, \epsilon_{i+1})$.

(7) Show that H_1 is abelian provided ϵ_1 is small enough with a bound depending only on n. (Hint: It suffices if $f(\epsilon_1, \epsilon_1) < \epsilon_1$.)

(8) Show that H_2 is nilpotent provided ϵ_2 is small enough with a bound depending only on n. (Hint: it suffices if $f(\epsilon_1, \epsilon_2) < \epsilon_1$ and $f(\epsilon_2, \epsilon_2) < \epsilon_2$.)

(9) Briefly indicate why the arguments above analogously prove that H_t is nilpotent provided ϵ_t is small enough with a bound depending only on n.

Remark. This exercise is a start towards proving Jordan's theorem about finite subgroups of $GL_n(\mathbf{C})$ having "large" commutative subgroups.