Problem set 6 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Let $G$ be a finite group. Prove the following are equivalent

$(1)$ $G$ is a simple group, and

$(2)$ for every non-trivial irreducible character $\chi$ of $G$ and all $g \in G$, $g \neq 1$ we have $\chi(g) \neq \chi(1)$.

Exercise 2. Let $H$ and $G$ be finite groups. Let $f_1, f_2 : H \to G$ be two group homomorphisms. If $(V, \pi)$ is a representation of $G$, then restricting along $f_1$ we get a representation $(V, \pi \circ f_1)$ of $H$ and restricting along $f_2$ we get a representation $(V, \pi \circ f_2)$ of $H$.

$(1)$ Give an example to show that it may happen that $(V, \pi \circ f_1)$ of $H$ and $(V, \pi \circ f_2)$ are not isomorphic as $H$-representations.

$(2)$ Suppose that there exists a $g \in G$ such that $f_1(h) = gf_2(h)g^{-1}$ for all $h \in H$. Show that $(V, \pi \circ f_1)$ of $H$ and $(V, \pi \circ f_2)$ are isomorphic as $H$-representations.

Exercise 3. Let $H \subset G$ be a subgroup of a finite group $G$. Let $(W, \rho)$ be a permutation representation of $H$. Show that $\text{Ind}^G_H(W)$ is a permutation representation of $G$.

Exercise 4. Let $G$ be a finite group. Let $(V, \pi)$ be a representation of $G$. We say $(V, \pi)$ is a monomial representation if there exists a basis $e_1, \ldots, e_n$ of $V$ such that for each $g \in G$ the map $\pi(g)$ permutes the $e_i$ up to scalars. In other words, for each $g \in G$ and $1 \leq i \leq n$ there exists a $j \in \{1, \ldots, n\}$ and a $\lambda \in \mathbb{C}^*$ such that $\pi(g)(e_i) = \lambda e_j$. Show that a monomial representation is a direct sum of representations of the form $\text{Ind}^G_H(\chi)$ where $H \subset G$ is a subgroup and $\chi : H \to \mathbb{C}^*$ is a character.