Problem set 7 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

**Exercise 1.** Let $G$ be a finite group. Let $V$ be a finite dimensional $\mathbb{C}$-vector space. Let

$$\pi_t : G \to GL(V), \quad t \in [0, 1]$$

be a continuous family of representations of $G$. This means that for each $t \in [0, 1]$ the map $\pi_t$ is a homomorphism and that for each fixed $g \in G$ the function $[0, 1] \to GL(V), \ t \mapsto \pi_t(g)$ is continuous. Prove that $\pi_0$ and $\pi_1$ are isomorphic representations of $G$.

**Exercise 2.** Let $p$ be a prime number. Let $G$ be the group

$$G = \langle a_1, a_2, b_1, b_2, c \rangle / \text{relations}$$

where the relations are

1. all of the elements $a_1, a_2, b_1, b_2, c$ have order $p$,
2. $c$ is central,
3. the pairs $a_1, a_2$ and $a_1, b_2$ and $a_2, b_1$ and $b_1, b_2$ commute,
4. $a_1 b_1 a_1^{-1} b_1^{-1} = c$ and $a_2 b_2 a_2^{-1} b_2^{-1} = c$.

You may use that in this group every element has a unique expression of the form $a_1^{i_1} a_2^{i_2} b_1^{j_1} b_2^{j_2} c^k$ with $0 \leq i_1, i_2, j_1, j_2, k < p$. Construct all irreducible representations of $G$ using the method discussed in the lectures.

**Exercise 3.** Let $p$ be an odd prime number. Let $k = \mathbb{F}_p$ be the field with $p$ elements. Let $G = GL_2(k)$. Let $B \subset G$ be the Borel subgroup, i.e.,

$$B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

Choose a character $\chi_0 : k^* \to \mathbb{C}^*$ and define a character $\chi : B \to \mathbb{C}^*$ by the rule

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_0(a)$$

Compute the character of $\text{Ind}_B^G(\chi)$.

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1 Here $[0, 1] \subset \mathbb{R}$ is the closed interval. Choosing a basis of $V$ we may think of $\pi_t$ as a homomorphism into $GL_n(\mathbb{C})$ where $n = \dim(V)$, so the condition is that the map $[0, 1] \to GL_n(\mathbb{C}), \ t \mapsto \pi_t(g)$ is continuous, which in turn means that the matrix coefficients $\pi_t(g)_{ij}$ are continuous maps $[0, 1] \to \mathbb{C}$. 