Problem set 8 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Let $G$ be a finite group. Recall that given an class function $f$ on $G$ we let $\psi_2(f)$ be the class function sending $g$ to $f(g^2)$. Let $(V, \pi)$ be a representation of $G$. Recall that in the proof of Theorem 3.2 in the notes on real representations we saw that

$$\psi_2(\chi_V) = \chi_{\text{Sym}^2(V)} - \chi_{\wedge^2(V)}$$

(1) What does it mean in terms of representations if $\psi_2(\chi_V) = \chi_W$ for some representation $(W, \rho)$ of $G$?

(2) Give an example of a finite group $G$ and an irreducible $(V, \pi)$ of dimension $> 1$ such that there exists a $(W, \rho)$ as in (1).

(3) Give an example of a finite group $G$ and an irreducible $(V, \pi)$ of dimension $> 1$ where there does not exist a $(W, \rho)$ as in (1).

(4) Show that if for every irreducible $(V, \pi)$ there exists a $(W, \rho)$ as in (1), then the same is true for every representation $(V, \pi)$ of $G$.

(5) For a class function $f$ on $G$ define $\psi_3(f)$ to be the class function sending $g$ to $f(g^3)$. Can you find an expression for $\psi_3(\chi_V)$ in terms of characters of representations of $G$?

Exercise 2. Let $a_0, a_1, \ldots$ be a series of integers. Suppose that there exist integers $c, n > 0$ such that for all $i = 0, 1, \ldots, n - 1$ there exists a polynomial $P_i \in \mathbb{Q}[x]$ with $a_i + nk = P_i(k)$ for $k \geq c$. Explain why $\sum_k a_k t^k$ is a rational function of $t$.

Remark. Let $A$ be a commutative, graded $C$-algebra which is finitely generated as a $C$-algebra and such that $\dim_C A_0 < \infty$. Then the sequence of numbers $a_k = \dim_C A_k$ satisfies the assumptions and hence the conclusions of Exercise 2 above.

Exercise 3. Consider the commutative, graded $C$-algebra

$$A = C[x_1, \ldots, x_r]/(f)$$

where $x_1, \ldots, x_r$ are homogenous of degrees $d_1, \ldots, d_r$ and $f$ is nonzero and homogenous of degree $e$. Compute the Poincaré series $P(A, t) = \sum_k \dim_C A_k$ as a rational function of $t$. An example is $C[x, y, z]/(x^{15} + y^{10} + z^6)$ where $\deg(x) = 2$, $\deg(y) = 3$, and $\deg(z) = 5$. Feel free to only work out the exercise in the example case.

Exercise 4. Let $G$ be a finite group. Let us consider the condition on the group

(*) for every representation $(V, \pi)$ of $G$ there exists a representation $(W, \rho)$ of $G$ such that $\psi_2(\chi_V) = \chi_W$.

We studied this condition in Exercise 1. My guess while writing these exercises is that (*) holds if $|G|$ is odd. Can you say anything about this? For example, take the group $G$ of order $p^5$ we considered in the previous exercise set (or you can take the similar group of order $p^3$ we discussed in the lecture that I called a "Heisenberg" group). Does (*) hold for $G$?