Problem set 8 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Let \( p \) be an odd prime number. Let \( k = \mathbb{F}_p \) be the field with \( p \) elements. Let \( G = GL_2(k) \). Let \( \epsilon \in k \) be a nonsquare (in particular \( \epsilon \neq 0 \); for example if \( p = 3, 7, 11 \) we can take \( \epsilon = -1 \)). Let \( H \subset G \) be the subgroup

\[
H = \left\{ \begin{pmatrix} a & \epsilon b \\ b & a \end{pmatrix} \mid (a, b) \neq (0, 0) \right\}
\]

Choose a nontrivial character \( \chi : H^* \to \mathbb{C}^* \). Compute the character of \( \text{Ind}^G_H(\chi) \).

Exercise 2. Let \( V \) be a two dimensional complex vector space. Let \( G \subset GL_2(V) \) be a finite subgroup. We proved in class that we may assume there is a hermitian positive definite form \( \langle \cdot, \cdot \rangle \) which is \( G \)-invariant. After choosing an orthonormal basis we may (and do) assume \( V = \mathbb{C}^2 \) with standard hermitian inner product and \( G \) is contained in the unitary \( 2 \times 2 \) matrices. Let \( s \in G \) be a complex reflection and let \( V = L \oplus M \) be the decomposition into \( s \)-eigenspaces. Given \( g \in GL(V) \) we say \( g \) fixes \( L \) and \( M \) if \( g(L) = L \) and \( g(M) = M \) and we say \( g \) switches \( L \) and \( M \) if \( g(L) = M \) and \( g(M) = L \).

(1) Prove that an element \( g \in GL(V) \) commutes with \( s \) if and only if \( g \) fixes \( L \) and \( M \).

(2) Let \( g \in GL(V) \) and set \( s' = gsg^{-1} \). Prove that if \( s' \) fixes \( L \) and \( M \), then \( g \) fixes or switches \( L \) and \( M \).

(3) Let \( g \in GL(V) \) and set \( g' = gsg^{-1}s^{-1} \). Prove that if \( g' \) fixes \( L \) and \( M \), then \( g \) fixes or switches \( L \) and \( M \).

(4) Let \( g \in GL(V) \) and set \( s' = gsg^{-1} \). If \( s \) has order \( m > 2 \), then prove that \( s' \) cannot switch \( L \) and \( M \).

(5) Let \( g \in GL(V) \) and set \( g' = gsg^{-1}s^{-1} \). If \( s \) has order \( m > 2 \), then prove that \( g' \) cannot switch \( L \) and \( M \).

(6) Assume \( s \) has order \( m > 2 \) and for all \( g \in G \) the element \( s' = gsg^{-1} \) fixes or switches \( L \) and \( M \). Prove that \( V \) is a monomial representation of \( G \).

(7) Let \( s \in G \) be a complex reflection with eigenvalues \( \lambda, 1 \). Prove that

\[
|s - 1| \leq |\lambda - 1|
\]

where on the left hand side we use the notation \( |A| \) for \( A \in GL_2(\mathbb{C}) \) is as in Exercise 3 of problem set 5. Hint: show that \( s \) can be written as \( u \text{diag}(\lambda, 1)u^{-1} \) for some unitary matrix \( u \) and then compute.

(8) Using the methods/results of Exercise 3 of problem set 5 and parts (1), (2) above, prove the following: if \( G \) has a complex reflection of “large” order \( m \), then \( V \) is a monomial representation. (This is a bit tricky. My calculation says this works for \( m > 53 \) which is very far from the truth, but I think it illustrates the idea.)

Exercise 3. Give an example of a finite group \( G \) and a homomorphism \( \pi : G \to GL_n(V) \) where \( V \) is a finite dimensional vector space over a field \( k \) of positive characteristic \( p \) such that \( (V, \pi) \) is not a direct sum of irreducible representations over the field \( k \).

Exercise 4. Let \( k \) be a field of characteristic \( p > 0 \). Let \( G \) be a finite group of order prime to \( p \). Let \( (V, \pi) \) be a representation of \( G \) over \( k \), i.e., \( V \) is a finite
dimensional $k$-vector space and $\pi : G \to GL(V)$ is a homomorphism. Show that

$$V^G = \{ x \in V \mid g(v) = v \text{ for all } g \in G \}$$

is the image of the projector

$$P = \frac{1}{|G|} \sum_{g \in G} \pi(g) \in \text{End}(V)$$

Deduce that if $(W, \rho)$ is a second representation of $G$ over $k$ and $V \to W$ is a surjective map of $G$-representations, then the induced map $V^G \to W^G$ is surjective too.

**Remark.** This exercise can be used to show that the category of representations of $G$ over $k$ is semi-simple, i.e., every representation is completely reducible (a direct sum of irreducible ones). Namely, if $V \to W$ is a surjective map of representations, then one looks at the surjective map of representations $V \otimes_k W^\vee \to W \otimes_k W^\vee$ and arguing that this induces a surjection on $G$-invariants, one finds a map of representations $W \to V$ whose composition with the given map $V \to W$ is the identity. Hence sub or quotient representations always split off, which implies complete reducibility as in the lectures.