Problem set 8 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Let p be an odd prime number. Let $k = \mathbf{F}_p$ be the field with p elements. Let $G = GL_2(k)$. Let $\epsilon \in k$ be a nonsquare (in particular $\epsilon \neq 0$; for example if $p = 3, 7, 11$ we can take $\epsilon = -1$). Let $H \subset G$ be the subgroup

$$H = \left\{ \begin{pmatrix} a & b\epsilon \\ b & a \end{pmatrix} \mid (a, b) \neq (0, 0) \right\}$$

Choose a nontrivial character $\chi : H^* \to \mathbf{C}^*$. Compute the character of $\mathrm{Ind}_H^G(\chi)$.

Exercise 2. Let V be a two dimensional complex vector space. Let $G \subset GL_2(V)$ be a finite subgroup. We proved in class that we may assume there is a hermitian positive definite form H which is G-invariant. After choosing an orthonormal basis we may (and do) assume $V = \mathbf{C}^2$ with standard hermitian inner product and G is contained in the unitary 2×2 matrices. Let $s \in G$ be a complex reflection and let $V = L \oplus M$ be the decomposition into s-eigenspaces. Given $g \in GL(V)$ we say *g fixes L and M* if $g(L) = L$ and $g(M) = M$ and we say *g switches L and M* if $g(L) = M$ and $g(M) = L$.

(1) Prove that an element $g \in GL(V)$ commutes with s if and only if g fixes L and M.
(2) Let $g \in GL(V)$ and set $s' = gsg^{-1}$. Prove that if s' fixes L and M, then g fixes or switches L and M.
(3) Let $g \in GL(V)$ and set $g' = gsg^{-1}s^{-1}$. Prove that if g' fixes L and M, then g fixes or switches L and M.
(4) Let $g \in GL(V)$ and set $s' = gsg^{-1}$. If s has order $m > 2$, then prove that s' cannot switch L and M.
(5) Let $g \in GL(V)$ and set $g' = gsg^{-1}s^{-1}$. If s has order $m > 2$, then prove that g' cannot switch L and M.
(6) Assume s has order $m > 2$ and for all $g \in G$ the element $s' = gsg^{-1}$ fixes or switches L and M. Prove that V is a monomial representation of G.
(7) Let $s \in G$ be a complex reflection with eigenvalues $\lambda, 1$. Prove that

$$|s - 1| \leq |\lambda - 1|$$

where on the left hand side we use the notation $|A|$ for $A \in GL_2(\mathbf{C})$ is as in Exercise 3 of problem set 5. Hint: show that s can be written as $u\mathrm{diag}(\lambda, 1)u^{-1}$ for some unitary matrix u and then compute.
(8) Using the methods/results of Exercise 3 of problem set 5 and parts (1), (2) above, prove the following: if G has a complex reflection of "large" order m, then V is a monomial representation. (This is a bit tricky. My calculation says this works for $m > 53$ which is very far from the truth, but I think it illustrates the idea.)

Exercise 3. Give an example of a finite group G and a homomorphism $\pi : G \to GL_n(V)$ where V is a finite dimensional vector space over a field k of positive characteristic p such that (V, π) is not a direct sum of irreducible representations over the field k.

Exercise 4. Let k be a field of characteristic $p > 0$. Let G be a finite group of order prime to p. Let (V, π) be a representation of G over k, i.e., V is a finite dimensional k-vector space and $\pi : G \to GL(V)$ is a homomorphism. Show that

$$V^G = \{x \in V \mid g(v) = v \text{ for all } g \in G\}$$

is the image of the projector

$$P = \frac{1}{|G|} \sum\nolimits_{g \in G} \pi(g) \in \mathrm{End}(V)$$

Deduce that if (W, ρ) is a second representation of G over k and $V \to W$ is a surjective map of G-representations, then the induced map $V^G \to W^G$ is surjective too.

Remark. This exercise can be used to show that the category of representations of G over k is semi-simple, i.e., every representation is completely reducible (a direct sum of irreducible ones). Namely, if $V \to W$ is a surjective map of representations, then one looks at the surjective map of representations $V \otimes_k W^\vee \to W \otimes_k W^\vee$ and arguing that this induces a surjection on G-invariants, one finds a map of representations $W \to V$ whose composition with the given map $V \to W$ is the identity. Hence sub or quotient representations always split off, which implies complete reducibility as in the lectures.