Real representations

1 Definition of a real representation

Definition 1.1. Let $V_\mathbb{R}$ be a finite dimensional real vector space. A real representation of a group $G$ is a homomorphism $\rho_{V_\mathbb{R}} : G \to \text{Aut} V_\mathbb{R}$, where $\text{Aut} V_\mathbb{R}$ denotes the $\mathbb{R}$-linear isomorphisms from $V_\mathbb{R}$ to itself. Homomorphisms and isomorphisms of real representations are defined in the obvious way. After a choice of basis, a real representation is equivalent to a homomorphism $\rho : G \to \text{GL}(n, \mathbb{R})$, and two such homomorphisms $\rho_1$ and $\rho_2$ are isomorphic real representations $\iff$ they are conjugate in $\text{GL}(n, \mathbb{R})$, i.e. $\iff$ there exists an $A \in \text{GL}(n, \mathbb{R})$ such that $\rho_2(g) = A\rho_1(g)A^{-1}$ for all $g \in G$.

Because $\text{GL}(n, \mathbb{R})$ is a subgroup of $\text{GL}(n, \mathbb{C})$, every real representation $V_\mathbb{R}$ defines a (complex) representation $V$. More abstractly, given a real vector space $V_\mathbb{R}$, we define its complexification to be the tensor product $V = V_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$. Concretely, think of $\mathbb{R}^n$ being enlarged to $\mathbb{C}^n$. For any real vector space $V_\mathbb{R}$, if $v_1, \ldots, v_n$ is a basis for $V_\mathbb{R}$, then by definition $V_\mathbb{R}$ is the set of all linear combinations of the $v_i$ with real coefficients, and $V$ is the set of all linear combinations of the $v_i$ with complex coefficients. In particular, $v_1, \ldots, v_n$ is a basis for the complex vector space $V$. Conversely, given a (complex) vector space $V$ and a basis $v_1, \ldots, v_n$ of $V$, we can define a vector subspace $V_\mathbb{R}$ of $V$ by taking the set of all linear combinations of the $v_i$ with real coefficients, and $V$ is then the complexification of $V_\mathbb{R}$. We have an inclusion $\text{Aut} V_\mathbb{R} \to \text{Aut} V$, which can be summarized by the commutative diagram

$$
\begin{array}{ccc}
\text{Aut} V_\mathbb{R} & \longrightarrow & \text{Aut} V \\
\cong \downarrow & & \downarrow \cong \\
\text{GL}(n, \mathbb{R}) & \longrightarrow & \text{GL}(n, \mathbb{C}),
\end{array}
$$

where the vertical isomorphisms correspond to the choice of basis $v_1, \ldots, v_n$. However, the top horizontal inclusion is canonical, i.e. does not depend on the choice of basis.
Definition 1.2. A representation $\rho_V : G \to \text{Aut } V$ can be defined over $\mathbb{R}$ if there exists a real vector space $V_\mathbb{R}$ and a real representation $\rho_{V_\mathbb{R}} : G \to \text{Aut } V_\mathbb{R}$ such that $V$ is the complexification of $V_\mathbb{R}$ and $\rho_V$ is the image of $\rho_{V_\mathbb{R}}$ via the inclusion $\text{Aut } V_\mathbb{R} \to \text{Aut } V$. Equivalently, there exists a basis of $V$ such that, for every $g \in G$, the matrices $\rho_V(g)$ have real entries.

Remark 1.3. (1) We can make the same definition for any subfield $K$ of $\mathbb{C}$, for example for $K = \mathbb{Q}$.

(2) Every complex vector space $V$ of dimension $n$ is also a real vector space of dimension $2n$, by only allowing scalar multiplication by real numbers. To see the statement about the dimensions, if $v_1, \ldots, v_n$ is a basis of $V$ as a complex vector space, then it is easy to check that $v_1, iv_1, v_2, iv_2, \ldots, v_n, iv_n$ is a basis of $V$ viewed as a real vector space. Since every complex linear isomorphism is automatically real linear, there is a homomorphism $\text{GL}(n, \mathbb{C}) \to \text{GL}(2n, \mathbb{R})$ which is a little messy to write down in general. For $n = 1$, it corresponds to the homomorphism $\varphi : \mathbb{C}^* \to \text{GL}(2, \mathbb{R})$ defined by $\varphi(a + bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

The following gives a necessary condition for a representation of a finite group to be defined over $\mathbb{R}$:

Lemma 1.4. If $\rho_V$ can be defined over $\mathbb{R}$, then, for all $g \in G$, $\chi_V(g) \in \mathbb{R}$. More generally, if $K$ is a subfield of $\mathbb{C}$ and $\rho_V$ can be defined over $K$, then, for all $g \in G$, $\chi_V(g) \in K$.

Proof. This is clear since the trace of an $n \times n$ matrix with entries in $K$ is an element of $K$.

As we shall see, the necessary condition above is not in general sufficient.

2 When is an irreducible representation defined over $\mathbb{R}$

We begin by analyzing the condition that $\chi_V(g) \in \mathbb{R}$ for all $g \in G$, and shall only consider the case of an irreducible representation in what follows.

Lemma 2.1. If $V$ is a $G$-representation of the finite group $G$, then $\chi_V(g) \in \mathbb{R}$ for all $g \in G \iff V \cong V^*$. If moreover $V$ is irreducible and $V \cong V^*$, then there exists a nonzero $\varphi \in \text{Hom}^G(V, V^*)$ and it is unique up to a nonzero scalar, i.e. $\dim \text{Hom}^G(V, V^*) = 1$. 

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Proof. Since $\chi_{V^*} = \overline{\chi_V}$, we see that $\chi_V(g) \in \mathbb{R}$ for all $g \in G$ $\iff$ $\chi_V = \overline{\chi_V}$ $\iff$ $\chi_V = \chi_{V^*}$ $\iff$ $V \cong V^*$. The remaining statement then follows from Schur’s lemma.

We define $\text{Bil}(V)$ to be the set of bilinear functions $F : V \times V \to \mathbb{C}$. General results about tensor products tell us that $\text{Bil}(V) \cong (V \otimes V)^* \cong V^* \otimes V^* \cong \text{Hom}(V, V^*)$.

However, we will explicitly construct the isomorphism $\text{Bil}(V) \cong \text{Hom}(V, V^*)$:

Lemma 2.2. The map $A : \text{Bil}(V) \to \text{Hom}(V, V^*)$ defined by

$$A(F)(v)(w) = F(v, w)$$

is an isomorphism of vector spaces. If $V$ is a $G$-representation and we define

$$\rho_{\text{Bil}(V)}(g)(F)(v, w) = F(\rho_V(g)^{-1}(v), \rho_V(g)^{-1}(w)),$$

then $A$ is a $G$-isomorphism, where as usual, given $\phi \in \text{Hom}(V, V^*)$,

$$\rho_{\text{Hom}(V, V^*)}(g)(\phi)(v)(w) = \phi(\rho_V(g)^{-1}v)(\rho_V(g)^{-1}w).$$

Proof. If we define $A(F)$ as in the statement, then it is easy to see that $A(F)(v)$ is linear in $w$ and that $v \mapsto A(F)(v)$ is linear in $v$, so that $A(F) \in \text{Hom}(V, V^*)$. Also, a short computation shows that $A(F_1 + F_2) = A(F_1) + A(F_2)$ and that $A(tF) = tA(F)$, so $A$ is a linear map of vector spaces. To show that $A$ is an isomorphism, we define an inverse function: let $B : \text{Hom}(V, V^*) \to \text{Bil}(V)$ be defined by

$$B(\phi)(v, w) = \phi(v)(w).$$

Again, an easy calculation shows that $B \circ A = \text{Id}$, $A \circ B = \text{Id}$. Finally, if $V$ is a $G$-representation, then

$$A(\rho_{\text{Bil}(V)}(g)(F))(v)(w) = \rho_{\text{Bil}(V)}(g)(F)(v, w) = F(\rho_V(g)^{-1}(v), \rho_V(g)^{-1}(w))$$

$$= \rho_{\text{Hom}(V, V^*)}(g)(A(F))(v)(w),$$

so that $A$ is a $G$-morphism and hence a $G$-isomorphism.

Corollary 2.3. If $V$ is an irreducible $G$-representation, then $\dim(\text{Bil}(V)^G)$ is 0 if $V$ is not isomorphic to $V^*$ and 1 if $V \cong V^*$. 

To analyze $\text{Bil}(V)$ further, we make the following definition:
Definition 2.4. Let $F \in \text{Bil}(V)$. Then $F$ is symmetric if $F(v, w) = F(w, v)$ for all $v, w \in V$, and $F$ is antisymmetric if $F(v, w) = -F(w, v)$ for all $v, w \in V$. Let $\text{Sym}^2 V^*$ be the set of all symmetric $F \in \text{Bil}(V)$ and let $\bigwedge^2 V^*$ denote the set of all antisymmetric $F \in \text{Bil}(V)$. Clearly both $\text{Sym}^2 V^*$ and $\bigwedge^2 V^*$ are vector subspaces of $\text{Bil}(V)$. If $V$ is a $G$-representation, so that $\text{Bil}(V)$ is also a $G$-representation, then from the definition of $\rho_{\text{Bil}(V)}$ it is easy to see that $\text{Sym}^2 V^*$ and $\bigwedge^2 V^*$ are $G$-invariant subspaces of $\text{Bil}(V)$.

Lemma 2.5. $\text{Bil}(V) = \text{Sym}^2 V^* \oplus \bigwedge^2 V^*$. If $V$ is a $G$-representation, then the above is a direct sum of $G$-invariant subspaces.

Proof. Define $\pi_1 : \text{Bil}(V) \to \text{Sym}^2 V^*$ and $\pi_2 : \text{Bil}(V) \to \bigwedge^2 V^*$ by:

$$\pi_1(F)(v, w) = \frac{1}{2}(F(v, w) + F(w, v));$$

$$\pi_2(F)(v, w) = \frac{1}{2}(F(v, w) - F(w, v)).$$

Then clearly $\pi_1(F) = F \iff F \in \text{Sym}^2 V^*$, $\pi_1(F) = 0 \iff F \in \bigwedge^2 V^*$, and similarly $\pi_2(F) = 0 \iff F \in \text{Sym}^2 V^*$, $\pi_2(F) = F \iff F \in \bigwedge^2 V^*$. Also $\pi_1 + \pi_2 = \text{Id}$. It then follows that $\text{Bil}(V) = \text{Sym}^2 V^* \oplus \bigwedge^2 V^*$. The last statement is then a general fact.

Corollary 2.6. Let $V$ be an irreducible representation. If $V$ and $V^*$ are not isomorphic, then $\text{Bil}(V)^G = 0$. If $V$ and $V^*$ are isomorphic, then either $\dim(\text{Sym}^2 V^*)^G = 1$ and $\bigwedge^2 V^* = 0$ or $\dim(\bigwedge^2 V^*)^G = 1$ and $\text{Sym}^2 V^* = 0$.

We can now state the main result concerning real representations:

Theorem 2.7. Let $V$ be an irreducible $G$-representation.

(i) $V$ and $V^*$ are not isomorphic $\iff \text{Bil}(V)^G = 0$.

(ii) $V \cong V^*$ and $V$ is defined over $\mathbb{R}$ $\iff \dim(\text{Sym}^2 V^*)^G = 1$ and $\bigwedge^2 V^* = 0$.

(iii) $V \cong V^*$ and $V$ is not defined over $\mathbb{R}$ $\iff \dim(\bigwedge^2 V^*)^G = 1$ and $\text{Sym}^2 V^* = 0$. Moreover, in this case $\dim V$ is even.

Proof. We have already seen (i). Also (ii) $\implies$ (iii), except for the last statement about the dimension, since (iii) is just the equivalence of the negations of the two statements of (ii). So we must prove (ii).
Suppose that $V$ is defined over $\mathbb{R}$. In other words, there exists a basis $v_1, \ldots, v_n$ of $V$ such that the matrix of $\rho_V$ with respect to this basis has real entries. Let $V_\mathbb{R}$ be the real span of the $v_i$:

$$V_\mathbb{R} = \left\{ \sum_{i=1}^n t_i v_i : t_i \in \mathbb{R} \right\}.$$ 

Thus $V_\mathbb{R}$ is a real vector subspace of $V$ and $\rho_V$ comes from a real representation $\rho_{V_\mathbb{R}}$ of $G$ on $V_\mathbb{R}$, i.e. $\rho_V$ is defined over $\mathbb{R}$. There exists a positive definite inner product (i.e. a symmetric $\mathbb{R}$-bilinear function) on $V_\mathbb{R}$, for example we could define

$$\langle \sum_{i=1}^n s_i v_i, \sum_{i=1}^n t_i v_i \rangle = \sum_{i=1}^n s_i t_i.$$ 

This inner product is not $G$-invariant, but we can make it $G$-invariant by averaging over $G$: define

$$F_\mathbb{R}(v, w) = \frac{1}{\#(G)} \sum_{g \in G} \langle \rho_{V_\mathbb{R}}(v), \rho_{V_\mathbb{R}}(w) \rangle.$$ 

Then $F_\mathbb{R}$ is symmetric and it is positive definite, because it is a sum of positive definite inner products. In particular $F_\mathbb{R} \neq 0$. Note that $F_\mathbb{R}$ is specified by its values $F_\mathbb{R}(v_i, v_j)$ and the $G$-invariance of $F_\mathbb{R}$ is equivalent to the statement that, for all $i, j$ and all $g \in G$,

$$F_\mathbb{R}(\rho_{V_\mathbb{R}}(v_i), \rho_{V_\mathbb{R}}(v_j)) = F_\mathbb{R}(v_i, v_j).$$

Now we can extend $F_\mathbb{R}$ to a $\mathbb{C}$-bilinear function $F$ on $V$, by defining

$$F(v, w) = \sum_{i,j} s_i t_j F_\mathbb{R}(v_i, v_j),$$

where $v = \sum_i s_i v_i$ and $w = \sum_i t_i w_i$. In particular, if $v, w \in V_\mathbb{R}$, then $F(v, w) = F_\mathbb{R}(v, w)$ so that $F \neq 0$. Moreover $F$ is symmetric because $F_\mathbb{R}$ is symmetric, and hence $F_\mathbb{R}(v_i, v_j) = F_\mathbb{R}(v_j, v_i)$. Finally, one checks that $F$ is $G$-invariant, which is equivalent to the statement that $F(\rho_{V_\mathbb{R}}(v_i), \rho_{V_\mathbb{R}}(v_j)) = F(v_i, v_j)$ and thus follows from the corresponding statement for $F_\mathbb{R}$. Thus $F$ is a nonzero element of $\Sym^2 V^* G$. It follows that $\dim(\Sym^2 V^*)^G = 1$ and $\bigwedge^2 V^* = 0$.

$\implies$ : We must use the existence of a nonzero $F \in \Sym^2 V^* G$ to show that $V$ is defined over $\mathbb{R}$. We begin with a digression on complex structures. Let $V_\mathbb{R}$ be a real vector space with complexification $V$. We can think of this
as follows: there exists a basis $v_1, \ldots, v_n$ of $V$ such that $V_{\mathbb{R}}$ is the real span of $v_1, \ldots, v_n$. Thus $v_1, iv_1, v_2, iv_2, \ldots, v_n, iv_n$ is a real basis of $V$ and every $v \in V$ can be uniquely written as $w + iu$, where $w, u \in V_{\mathbb{R}}$. In other words, as real vector spaces,

$$V \cong V_{\mathbb{R}} \oplus iV_{\mathbb{R}}.$$  

Now we can define conjugation $\gamma$ on $V$:

$$\gamma(w + iu) = w - iu.$$  

Thus $\gamma$ is $\mathbb{R}$-linear with $\gamma^2 = \text{Id}$ and $+1$-eigenspace $V_{\mathbb{R}}$ and $-1$-eigenspace $iV_{\mathbb{R}}$. In terms of the basis $v_1, \ldots, v_n$, if $v = \sum_i t_i v_i$ is a vector in $V$, then

$$\gamma(\sum_i t_i v_i) = \sum_i \bar{t}_i v_i.$$  

Then $\gamma$ is conjugate linear: $\gamma(tv) = \bar{t}\gamma(v)$. Finally, if $A \in \text{End} V$ satisfies $A(V_{\mathbb{R}}) \subseteq V_{\mathbb{R}}$, i.e. the matrix of $A$ with respect to the basis $v_1, \ldots, v_n$ has real entries, then $A$ commutes with $\gamma$ since it preserves two eigenspaces, and conversely, if $A$ commutes with $\gamma$, then $A(V_{\mathbb{R}}) \subseteq V_{\mathbb{R}}$ and hence the matrix of $A$ with respect to the basis $v_1, \ldots, v_n$ has real entries.

Conversely, suppose that $V$ is a complex vector space and that $\gamma: V \to V$ is conjugate linear and hence $\mathbb{R}$-linear, and that $\gamma^2 = \text{Id}$. Then $\gamma$ is diagonalizable over $\mathbb{R}$, i.e. $V \cong V_+ \oplus V_-$, where $V_\pm$ are real vector subspaces of $V$, $\gamma|V_+ = \text{Id}$ and $\gamma|V_- = -\text{Id}$. In fact, we can define $V_+$ to be the $+1$-eigenspace of $\gamma$ and $V_-$ to be the $-1$-eigenspace. Then setting

$$\pi_+(v) = \frac{1}{2}(v + \gamma(v))$$

$$\pi_-(v) = \frac{1}{2}(v - \gamma(v)),$$

it is easy to check that $\text{Im} \pi_\pm = V_\pm$, $\text{Ker} \pi_\pm = V_\mp$, and $\pi_+ + \pi_- = \text{Id}$, giving the direct sum decomposition. Moreover

$$v \in V_+ \iff \gamma(v) = v \iff \gamma(iv) = -iv \iff iv \in V_-.$$  

Thus multiplication by $i$ defines an isomorphism from $V_+$ to $V_-$. It follows that an $\mathbb{R}$-basis for $V_+$ is a $\mathbb{C}$-basis for $V$, and that $V$ is the complexification of $V_+$,. Finally, if $A \in \text{End} V$ is complex linear and $A$ commutes with $\gamma$, then $A(V_+) \subseteq V_+$. Hence $A$ has real coefficients with respect to any basis of $V$ which is a real basis of $V_+$. In particular, if $\rho_V: G \to \text{Aut} V$ is a homomorphism and $\gamma$ commutes with $\rho_V(g)$ for every $g \in G$, then $\rho_V$ defines a real representation on $V_+$ and $\rho_V$ is the complexification of this representation.
Returning to our situation, we have a nonzero symmetric $G$-invariant $F \in \text{Sym}^2 V^*$, corresponding to a $G$-invariant homomorphism $\varphi: V \to V^*$, necessarily an isomorphism by Schur’s lemma. Here $F(v, w) = \varphi(v)(w)$, so the symmetry condition is the statement that, for all $v, w \in V$,

$$\varphi(v)(w) = \varphi(w)(v).$$

There exists a positive definite Hermitian inner product on $V$, so after averaging there exists a $G$-invariant positive definite Hermitian inner product $H(v, w)$ on $V$. Such an $H$ defines an $\mathbb{R}$-linear function $\psi: V \to V^*$ by the rule

$$\psi(v)(w) = H(w, v).$$

Note that we have to switch the order to make $\psi(v)$ is linear in $w$. However, $\psi$ is conjugate linear in $v$. It is easy to see that $\psi$ is an isomorphism: since $V$ and $V^*$ have the same dimension as real vector spaces, it suffices to show that $\psi$ is injective, i.e. that $\psi(v)$ is not the zero element in $V^*$ for $v \neq 0$. This follows since $\psi(v)(v) = H(v, v) > 0$.

Define $\alpha: V \to V$ by $\alpha = \psi^{-1} \circ \varphi$. Then $\alpha$ is conjugate linear since it is a composition of a complex linear and a conjugate linear map, and $\alpha$ is an isomorphism of real vector spaces. Finally, $\alpha$ is $G$-invariant since $\psi$ and $\varphi$ are $G$-invariant.

Consider $\alpha^2: V \to V$, which is complex linear as it is the composition of two conjugate linear maps. It is also a $G$-invariant isomorphism since it is the composition of two such. Thus, by Schur’s lemma, $\alpha^2 = \lambda \text{Id}$ for some nonzero complex number $\lambda$.

**Claim 2.8.** $\lambda$ is a positive real number.

**Proof.** By the definition of $\psi$, $\psi(v)(w) = H(v, w)$ for all $v, w \in V$. Thus, for all $f \in V^*$,

$$f(w) = H(w, \psi^{-1}(f)).$$

If in addition $f = \varphi(v)$, this says that

$$F(v, w) = \varphi(v)(w) = H(w, \psi^{-1} \circ \varphi(v)) = H(w, \alpha(v)).$$

Replacing $w$ by $\alpha(w)$ and using the symmetry of $F$ gives

$$H(\alpha(w), \alpha(v)) = F(v, \alpha(w)) = F(\alpha(w), v) = H(v, \alpha^2(w)) = H(v, \lambda w).$$

Now choose $v = w, v \neq 0$. We get

$$H(\alpha(v), \alpha(v)) = H(v, \lambda v) = \lambda H(v, v).$$
Since both $H(\alpha(v), \alpha(v))$ and $H(v, v)$ are real and positive, it follows that $\overline{\lambda}$ is real and positive, and thus the same is true for $\lambda = \overline{\lambda}$.

Returning to the proof of Theorem 2.7, define $\gamma = \lambda^{-1/2}\alpha$. Then $\gamma$ is a conjugate linear isomorphism and $\gamma^2 = \text{Id}$. Finally, $\gamma$ commutes with the $G$-action, and so as in the above discussion on real structures, $\gamma$ defines a real structure on $V$ for which $\rho_V$ is a real representation.

Thus we have proved all of the statements in Theorem 2.7 except for the fact that, in case (iii), $\dim V$ is even. This is a general linear algebra fact about vector spaces for which there exists an $F \in \wedge^2 V^*$ such that the corresponding map $V \to V^*$ is an isomorphism.

### 3 A computational characterization

We would like a computational method for deciding when a representation can be defined over $\mathbb{R}$. First, a definition:

**Definition 3.1.** Let $f$ be a class function on $G$. Define a new function $\psi_2(f)$ by

$$\psi_2(f)(g) = f(g^2).$$

Then $\psi_2(f)$ is also a class function, since

$$\psi_2(f)(xgx^{-1}) = f((xgx^{-1})^2) = f(xg^2x^{-1}) = f(g^2) = \psi_2(f)(g).$$

(One can define $\psi_n(f)$ similarly for every $n \in \mathbb{Z}$.)

In particular, for a character $\chi_V$ of $G$, we can consider the expression

$$\langle \psi_2(\chi_V), 1 \rangle = \frac{1}{\#(G)} \sum_{g \in G} \chi_V(g^2).$$

**Theorem 3.2.** Let $V$ be an irreducible $G$-representation.

(i) $V$ and $V^*$ are not isomorphic $\iff \langle \psi_2(\chi_V), 1 \rangle = 0$.

(ii) $V \cong V^*$ and $V$ is defined over $\mathbb{R} \iff \langle \psi_2(\chi_V), 1 \rangle = 1$.

(iii) $V \cong V^*$ and $V$ is not defined over $\mathbb{R} \iff \langle \psi_2(\chi_V), 1 \rangle = -1$.

**Proof.** By Theorem 2.7, we have the following:

(i) $V$ and $V^*$ are not isomorphic $\iff \text{Hom}^G(V, V^*) = \text{Bil}^G(V) = 0$

$\iff \langle \chi_{\text{Bil}^G(V)}, 1 \rangle = 0$. 

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(ii) $V \cong V^*$ and $V$ is defined over $\mathbb{R} \iff \dim(\text{Sym}^2 V^*) = 1$ and $\wedge^2 V^* = 0 \iff \langle \chi_{\text{Sym}^2 V^*}, 1 \rangle = 1$ and $\langle \chi_{\wedge^2 V^*}, 1 \rangle = 0$.

(iii) $V \cong V^*$ and $V$ is not defined over $\mathbb{R} \iff \dim(\wedge^2 V^*) = 1$ and $\text{Sym}^2 V^* = 0 \iff \langle \chi_{\wedge^2 V^*}, 1 \rangle = 1$ and $\langle \chi_{\text{Sym}^2 V^*}, 1 \rangle = 0$.

So we must compute these characters. In fact, we claim:

(i) $\chi_{\text{Bil}(V)} = \chi_{\text{Hom}(V, V^*)} = \bar{\chi}_V$.

(ii) $\chi_{\text{Sym}^2 V^*} = \frac{1}{2}(\bar{\chi}_V^2 + \psi_2(\bar{\chi}_V))$.

(iii) $\chi_{\wedge^2 V^*} = \frac{1}{2}(\bar{\chi}_V^2 - \psi_2(\bar{\chi}_V))$.

Assuming this, we have

$$\dim(\text{Sym}^2 V^*) = \langle \chi_{\text{Sym}^2 V^*}, 1 \rangle = \frac{1}{2}(\langle \bar{\chi}_V^2, 1 \rangle + \langle \psi_2(\bar{\chi}_V), 1 \rangle);$$

$$\dim(\wedge^2 V^*) = \langle \chi_{\wedge^2 V^*}, 1 \rangle = \frac{1}{2}(\langle \bar{\chi}_V^2, 1 \rangle - \langle \psi_2(\bar{\chi}_V), 1 \rangle).$$

Then $\langle \psi_2(\bar{\chi}_V), 1 \rangle = 0 \iff \dim(\text{Sym}^2 V^*) = \dim(\wedge^2 V^*)$, which happens exactly when $\text{Hom}(G, V^*) = 0$, since otherwise one of the dimensions is 0 and the other is 1. Since $\langle \psi_2(\bar{\chi}_V), 1 \rangle$ is real in this case,

$$\langle \psi_2(\bar{\chi}_V), 1 \rangle = \langle \psi_2(\bar{\chi}_V), 1 \rangle.$$

A brief computation in the remaining cases shows that $\langle \chi_{\text{Sym}^2 V^*}, 1 \rangle = 1$ and $\langle \chi_{\wedge^2 V^*}, 1 \rangle = 0 \iff \langle \psi_2(\bar{\chi}_V), 1 \rangle = 1$, and $\langle \chi_{\text{Sym}^2 V^*}, 1 \rangle = 0 \iff \langle \psi_2(\bar{\chi}_V), 1 \rangle = -1$. As before, by taking conjugates, the first case happens $\iff \langle \psi_2(\bar{\chi}_V), 1 \rangle = 1$ and the second $\iff \langle \psi_2(\bar{\chi}_V), 1 \rangle = -1$.

So we must prove the claim. For $g \in G$, the linear map $\rho_V(g)$ is diagonalizable. Let $v_1, \ldots, v_n$ be a basis of $V$ such that $\rho_V(v_i) = \lambda_i v_i$ and let $v_i^*$ be the dual basis. Then $\rho_V(g^2)$ is also diagonalized by the basis $v_1, \ldots, v_n$, with eigenvalues $\bar{\lambda}_i^2$, and hence

$$\psi_2(\bar{\chi}_V)(g) = \bar{\chi}_V(g^2) = \sum_i \bar{\lambda}_i^2.$$

Let $v_i v_j^* \in \text{Hom}(V, V^*)$ be the linear map defined by $v_i v_j^*(w) = v_i^*(w)v_j^*$. Then $v_i^* v_j^*, 1 \leq i, j \leq n$ is a basis for $\text{Hom}(V, V^*)$ and each $v_i^* v_j^*$ is an
eigenvector for $\rho_{\text{Hom}(V,V^*)}(g)$ with eigenvalue $\lambda_i^{-1}\lambda_j^{-1} = \bar{\lambda}_i\bar{\lambda}_j$. Thus we see as previously noted that

$$\chi_{\text{Hom}(V,V^*)}(g) = \sum_{i,j} \bar{\lambda}_i \bar{\lambda}_j = \left( \sum_i \bar{\lambda}_i \right) \left( \sum_j \bar{\lambda}_j \right) = (\bar{\chi}_V(g))^2.$$ 

We can also write this as

$$(\bar{\chi}_V(g))^2 = \sum_i \bar{\lambda}_i^2 + 2 \sum_{i < j} \bar{\lambda}_i \bar{\lambda}_j.$$ 

As for $\text{Sym}^2 V^*$, we can find a basis for it by symmetrizing the expressions $v_i^* v_j^*$ to $\frac{1}{2}(v_i^* v_j^* + v_j^* v_i^*)$. This expression is unchanged by switching $i$ and $i$, and the functions

$$\frac{1}{2}(v_i^* v_j^* + v_j^* v_i^*), \quad i \leq j$$

are linearly independent. A similar argument shows that a basis for $\Lambda^2 V^*$ is given by

$$\frac{1}{2}(v_i^* v_j^* - v_j^* v_i^*), \quad i < j.$$ 

Thus

$$\chi_{\text{Sym}^2 V^*}(g) = \sum_{i \leq j} \bar{\lambda}_i \bar{\lambda}_j;$$

$$\chi_{\Lambda^2 V^*}(g) = \sum_{i < j} \bar{\lambda}_i \bar{\lambda}_j.$$ 

Then

$$\frac{1}{2}(\bar{\chi}_V(g) + \psi_2(\bar{\chi}_V)(g)) = \frac{1}{2} \left( \sum_i \bar{\lambda}_i^2 + 2 \sum_{i < j} \bar{\lambda}_i \bar{\lambda}_j + \sum_i \bar{\lambda}_i^2 \right)$$

$$= \frac{1}{2} \left( 2 \sum_i \bar{\lambda}_i^2 + 2 \sum_{i < j} \bar{\lambda}_i \bar{\lambda}_j \right)$$

$$= \sum_{i \leq j} \bar{\lambda}_i \bar{\lambda}_j = \chi_{\text{Sym}^2 V^*}(g).$$

A similar calculation establishes the formula for $\chi_{\Lambda^2 V^*}(g)$.
Example 3.3. (1) Let $G = D_4$ and let $V$ be the irreducible 2-dimensional representation of $D_4$. The elements of $D_4$ are $\alpha^k$, $0 \leq k \leq 3$, and $\tau \alpha^k$, $0 \leq k \leq 3$. Moreover, $(\alpha^k)^2 = 1$ if $k = 0, 2$, $(\alpha^k)^2 = \alpha^2$ if $k = 1, 3$, and $(\tau \alpha^k)^2 = 1$ for all $k$. Thus, in $D_4$, there are 6 elements whose square is 1 and 2 elements whose square is $\alpha^2$. We know that $\chi_V(1) = 2$ and that $\chi_V(\alpha^2) = -2$. Then
\[
\langle \psi_2(\chi_V), 1 \rangle = \frac{1}{8}(6 \cdot 2 + 2 \cdot (-2)) = 1.
\]
Thus the irreducible 2-dimensional representation of $D_4$ can be defined over $\mathbb{R}$. Of course, we have seen this directly.

(2) Let $G = Q$, the quaternion group, and let $V$ be the irreducible 2-dimensional representation of $Q$. The elements of $Q$ are $\pm 1, \pm i, \pm j, \pm k$. Moreover, $(1)^2 = (-1)^2 = 1$ and all other elements have square $-1$. Thus, in $Q$, there are 2 elements whose square is 1 and 6 elements whose square is $-1$. We know that $\chi_V(1) = 2$ and that $\chi_V(-1) = -2$. Then
\[
\langle \psi_2(\chi_V), 1 \rangle = \frac{1}{8}(2 \cdot 2 + 6 \cdot (-2)) = -1.
\]
Thus the irreducible 2-dimensional representation of $Q$ cannot be defined over $\mathbb{R}$.

4 Irreducible real representations

In this section, we switch gears and look at things from the perspective of an irreducible real representation $V_\mathbb{R}$. We shall just state the main result (although its proof is not that difficult).

Recall that, if $V_\mathbb{R}$ is a real representation which is irreducible as a real representation, then Schur’s lemma only says that $\text{Hom}^G(V_\mathbb{R}, V_\mathbb{R})$ is a division ring containing $\mathbb{R}$ in its center, and is a finite dimensional $\mathbb{R}$-vector space since it is isomorphic to a vector subspace of $M_n(\mathbb{R})$ for $n = \dim V_\mathbb{R}$. It is not hard to classify such division rings: $\text{Hom}^G(V_\mathbb{R}, V_\mathbb{R})$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, the quaternions. We can then look at the complexification $V$ of $V_\mathbb{R}$. Although $V_\mathbb{R}$ is an irreducible real representation, $V$ need not be irreducible. The possibilities are as follows:

**Theorem 4.1.** Let $V_\mathbb{R}$ be an irreducible real representation and let $V$ be its complexification.

(i) $\text{Hom}^G(V_\mathbb{R}, V_\mathbb{R}) \cong \mathbb{R} \iff V$ is irreducible.
(ii) $\text{Hom}^G(V^*_R, V_R) \cong \mathbb{C} \iff V \cong W \oplus W^*$, where $W$ and $W^*$ are irreducible and $W$ and $W^*$ are not isomorphic.

(iii) $\text{Hom}^G(V^*_R, V_R) \cong \mathbb{H} \iff V \cong W \oplus W^*$, where $W$ is irreducible and $W \cong W^*$.

5 Real conjugacy classes

One question related to our previous discussion is the following: given a representation $V$ of $G$, when is $\chi_V(g) \in \mathbb{R}$? Of course, if $g$ is conjugate to $g^{-1}$ then, for every representation $V$,

$$\chi_V(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)},$$

and thus $\chi_V(g) \in \mathbb{R}$ for every $V$.

We make the following preliminary observation:

**Lemma 5.1.** Let $x \in G$, and suppose that $y$ is conjugate to $x$. Then $y^{-1}$ is conjugate to $x^{-1}$. Hence, $C(x)$ is the conjugacy class of $x$ and if we define

$$C(x)^{-1} = \{y^{-1} : y \in C(x)\},$$

then $C(x)^{-1} = C(x^{-1})$.

**Proof.** For the first statement, if $y = gxg^{-1}$, then

$$y^{-1} = (gxg^{-1})^{-1} = gx^{-1}g.$$

Thus, if $y \in C(x)$, then $y^{-1} \in C(x^{-1})$ and so $C(x)^{-1} \subseteq C(x^{-1})$. Conversely, if $z \in C(x^{-1})$, then $z$ is conjugate to $x^{-1}$ and hence $y = z^{-1}$ is conjugate to $(x^{-1})^{-1} = x$. Then by definition $z = y^{-1} \in C(x)^{-1}$, so that $C(x^{-1}) \subseteq C(x)^{-1}$. Thus $C(x)^{-1} = C(x^{-1})$. \qed

**Definition 5.2.** A conjugacy class $C(x)$ is real if $C(x)^{-1} = C(x)$, or equivalently if there exists a $y \in C(x)$ such that $y$ is conjugate to $y^{-1}$. By the lemma, if there exists one such $y$, then $y$ is conjugate to $y^{-1}$ for every $y \in C(x)$.

**Example 5.3.** (1) Clearly, $C(1) = \{1\}$ is a real conjugacy class.

(2) If $G$ is abelian, then $C(x) = \{x\}$ for every $x \in G$. Thus $C(x)$ is a real conjugacy class $\iff x = x^{-1} \iff x$ has order 1 or 2. In particular, if $G$ is an abelian group of odd order, then the only real conjugacy class is $C(1)$. 

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(3) In $S_n$, every element $\sigma$ is conjugate to $\sigma^{-1}$, and thus every conjugacy class is real. In fact, every element $\sigma$ can be written as $\sigma = \gamma_1 \cdots \gamma_\ell$, where each $\gamma_i$ is a cycle of some length $n_i > 1$ and the $\gamma_i$ are pairwise disjoint. As disjoint cycles commute,

$$\sigma^{-1} = (\gamma_1 \cdots \gamma_\ell)^{-1} = \gamma_\ell^{-1} \cdots \gamma_1^{-1} = \gamma_1^{-1} \cdots \gamma_\ell^{-1}.$$ 

But each $\gamma_i^{-1}$ is also a cycle of length $n_i$, and it is easy to check that, in $S_n$, two elements $\gamma_1 \cdots \gamma_\ell$ and $\delta_1 \cdots \delta_\ell$, both products of disjoint cycles of the same lengths, are conjugate.

(4) It is easy to check that the quaternion group $Q$ also has the property that every element $g$ is conjugate to $g^{-1}$, and thus that every conjugacy class is real.

As usual, enumerate the irreducible representations of $G$ up to isomorphism as $V_1, \ldots, V_h$. We then have the following curious fact about real conjugacy classes:

**Theorem 5.4 (Burnside).** The number of real conjugacy classes of $G$ is equal to the number of $i$ such that the irreducible representation $V_i$ is isomorphic to $V_i^*$, or equivalently such that $\chi_{V_i}(g) \in \mathbb{R}$ for all $g \in G$.

**Proof.** Enumerate the set of conjugacy classes of $G$ as $C(x_1), \ldots, C(x_h)$. Note that this enumeration doesn’t necessarily have anything to do with the enumeration $V_1, \ldots, V_h$ of irreducible representations chosen above. As usual, we let $Z \subseteq L^2(G)$ be the subspace of class functions. Then there are two natural bases for $Z$: the set of characteristic functions $f_{C(x_i)}$ and the set of characters $\chi_{V_i}$. We abbreviate $f_{C(x_i)}$ by $f_i$. There are two permutations $\tau$ and $\sigma$ of the index set $\{1, \ldots, h\}$. We let $\tau(i)$ be the unique $j$ such that $C(x_i)^{-1} = C(x_j)$, and we let $\sigma(i)$ be the unique $j$ such that $V_i^* = V_j$. The content of the theorem is then that the number of $i$ such that $\tau(i) = i$ is equal to the number of $i$ such that $\sigma(i) = i$.

As permutations of the index set, both $\tau$ and $\sigma$ define permutation matrices $P_\tau, P_\sigma \in GL(h, \mathbb{C})$ by the rule $P_\tau(f_i) = f_{\tau(i)}$, and similarly $P_\sigma(f_i) = f_{\sigma(i)}$. As with all permutation matrices, $\text{Tr } P_\tau$ is the number of $i$ such that $\tau(i) = i$ and $\text{Tr } P_\sigma$ is the number of $i$ such that $\sigma(i) = i$. So we must show that $\text{Tr } P_\tau = \text{Tr } P_\sigma$. It suffices to find an invertible $h \times h$ matrix $M$ such that $P_\tau \cdot M = M \cdot P_\sigma$, for then $P_\tau = MP_\sigma M^{-1}$ and so the traces are equal.

Let $M = (\chi_{V_j}(x_i))$. Then $M$ is the change of basis matrix for the two bases $f_1, \ldots, f_h$ and $\chi_{V_1}, \ldots, \chi_{V_h}$, because

$$M f_i = \sum_{j=1}^h \chi_{V_j}(x_j) f_j = \sum_{j=1}^h \chi_{V_i}(x_j) f_{C(x_j)},$$

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and by comparing the values of the above class function on every $x_j$ we see that $Mf_i = \chi_{V_i}$. In particular, $M$ is invertible. By definition, since $P_\tau$ acts by permuting the $f_i$ according to $\tau$, we see that

$$P_\tau Mf_i = \sum_{j=1}^{h} \chi_{V_i}(x_j)f_{\tau(j)} = \sum_{j=1}^{h} \chi_{V_i}(x_j)f_{C(x_j^{-1})}$$

$$= \sum_{j=1}^{h} \chi_{V_i}(x_j^{-1})f_{C(x_j)} = \sum_{j=1}^{h} \chi_{V_i^*}(x_j)f_{C(x_j)}$$

$$= \chi_{V_i^*}.$$

On the other hand,

$$MP_\sigma f_i = Mf_{\sigma(i)} = \chi_{V_{\sigma}(i)} = \chi_{V_i^*}.$$

Thus $P_\tau \cdot M = M \cdot P_\sigma$, concluding the proof. \qed

The following is a purely group-theoretic argument:

**Proposition 5.5.** The order $\#(G)$ is odd $\iff$ the only real conjugacy class is $C(1) = \{1\}$.

**Proof.** Equivalently, we have to show that the order $\#(G)$ is even $\iff$ there exists a real conjugacy class $C(x)$ with $x \neq 1$. If $\#(G)$ is even, then an easy special case of Cauchy’s theorem says that there exists an element $x$ of order 2. Then $x \neq 1$ and $x = x^{-1}$, so that $C(x)$ is a real conjugacy class and $C(x) \neq C(1)$.

Conversely, suppose that there exists a real conjugacy class $C(x)$ with $x \neq 1$. If $x = x^{-1}$, then $x$ has order 2. By Lagrange’s theorem, the order of any element of $G$ divides the order of $G$, so $\#(G)$ is even in this case. Otherwise, $x \neq x^{-1}$ but there exists an $h \in G$ such that $hxh^{-1} = x^{-1}$. Then $h^2xh^{-2} = h(hxh^{-1})h^{-1} = hx^{-1}h^{-1} = x$, and by induction we see that

$$h^a xh^{-a} = \begin{cases} x^{-1}, & \text{if } a \text{ is odd;} \\ x, & \text{if } a \text{ is even.} \end{cases}$$

Let $N$ be the order of $h$. Then $N$ must be even, since $h^N xh^{-N} = 1x1 = x$ and $x \neq x^{-1}$. But then $N$ divides $\#(G)$, again by Lagrange’s theorem, so that $\#(G)$ is divisible by an even number and hence is even. \qed

**Corollary 5.6.** The order $\#(G)$ is odd $\iff$ the only irreducible representation $V_i$ such that $V_i \cong V_i^*$, or equivalently such that $\chi_{V_i}(g) \in \mathbb{R}$ for all $g \in G$, is the trivial representation.
Proof. By the previous corollary, \( \#(G) \) is odd \iff there exists exactly one real conjugacy class \iff there exists exactly one irreducible representation \( V \) of \( G \) up to isomorphism such that \( V \cong V^* \). Since the trivial representation has this property, if there is only one such it must be the trivial representation.

We then have the following purely group-theoretic fact:

**Proposition 5.7.** If \( \#(G) \) is odd and the number of conjugacy classes of \( G \) is \( h \), then

\[
h \equiv \#(G) \pmod{16}.
\]

**Proof.** We use the following basic fact: if \( n \) is an odd integer, then \( n^2 \equiv 1 \pmod{8} \). This can be proved by checking all the possibilities (\( n \) must be \( \equiv 1, 3, 5, 7 \pmod{8} \)), or directly:

\[
n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 4m(m + 1) + 1.
\]

But \( m(m + 1) \) is always even, so \( n^2 \equiv 1 \pmod{8} \).

Now let \( h \) be the number of conjugacy classes of \( G \) or equivalently the number of irreducible representations up to isomorphism. If we enumerate these as \( V_1, \ldots, V_h \), we can assume that \( V_1 \) is the trivial representation. If \( d_i = \dim V_i \), then \( d_1 = 1 \) and \( d_i \) divides \( \#(G) \), hence \( d_i = 2e_i + 1 \). Moreover, if \( i \neq 1 \), then \( V_i \) and \( V_i^* \) are not isomorphic, so there are \( 2r \) remaining representations \( V_i \) and they occur in pairs \( V_i, V_i^* \) with \( \dim V_i^* = \dim V_i = 2e_i + 1 \). For example, we could index the representations so that \( V_1 \) is the trivial representation and \( V_{i+2} \cong V_i^* \), \( \ldots \), \( V_{2r+1} \cong V_{r+1}^* \). Hence \( d_i = d_{i+r} \) for \( 2 \leq i \leq r + 1 \). Then

\[
\#(G) = \sum_{i=1}^{h} d_i^2 = 1 + \sum_{i=2}^{2r+1} d_i^2 = 1 + 2 \sum_{i=2}^{r+1} d_i^2
\]

\[
= 1 + 2 \sum_{i=2}^{r+1} (2e_i + 1)^2 = 1 + \sum_{i=2}^{r+1} 2(4e_i^2 + 4e_i + 1)
\]

\[
= 1 + 8 \sum_{i=2}^{r+1} (e_i + 1) = 2r + 1 + \sum_{i=2}^{r+1} 8e_i(e_i + 1).
\]

As before, \( 8e_i(e_i + 1) \equiv 0 \pmod{16} \), so that

\[
\#(G) \equiv 2r + 1 = h \pmod{16}
\]

as claimed. \( \square \)
6 The case of the rational numbers

We begin with some general comments about the possible values of a character $\chi_V$ of a finite group $G$.

**Definition 6.1.** If $G$ is a finite group, then the **exponent** of $G$ is the least common multiple of the orders of the elements of $G$. Equivalently, $N$ is the smallest positive integer such that $g^N = 1$ for all $g \in G$. Note that $N$ divides $\#(G)$ and that (by Cauchy’s theorem) $N$ and $\#(G)$ have the same prime factors. However, $N$ can be strictly smaller than $\#(G)$. For example, for $D_4$, $N = 4$ but $\#(D_4) = 8$. More generally, if $\#(G) = p^n$ where $p$ is a prime, then the exponent of $G$ is $p^n \iff G$ is cyclic. For another example, the exponent of $S_4$ is 12 but $\#(S_4) = 24$.

Suppose now that $\rho_V$ is a $G$-representation. For every $g \in G$, the eigenvalues $\lambda_i$ of $\rho_V(g)$ are $a$th roots of unity, where $a$ is the order of $g$, and hence they are $N$th roots of unity, where $N$ is the exponent of $G$. Since $\chi_V(g)$ is the sum of the $\lambda_i$, it follows that, for every $g \in G$, $\chi_V(g) \in \mathbb{Q}(\mu_N)$, the extension of $\mathbb{Q}$ generated by the $N$th roots of unity.

We can then ask when a $G$-representation $V$ is defined over $\mathbb{Q}$. In general, this is a very hard question. An easy question to ask is: given a finite group $G$, when is $\chi_V(g) \in \mathbb{Q}$ for every representation $V$ of $G$ (or equivalently every irreducible representation) and every $g \in G$? Note that $\chi_V(g) \in \mathbb{Q} \iff \chi_V(g) \in \mathbb{Z}$, since $\chi_V(g)$ is an algebraic integer. This question can be answered:

**Theorem 6.2.** For a finite group $G$, $\chi_V(g) \in \mathbb{Q}$ for every representation $V$ of $G$ and every element $g \in G \iff$ for every $g \in G$ and every $a \in \mathbb{Z}$ such that $\gcd(a, \#(G)) = 1$, $g^a$ is conjugate to $g$.

**Proof.** We shall give the proof modulo a little Galois theory and number theory. Note that $\mathbb{Q}(\mu_N)$ is a normal, hence Galois extension of $\mathbb{Q}$ since it is the splitting field of $x^N - 1$. Thus, given $\alpha \in \mathbb{Q}(\mu_N)$, $\alpha \in \mathbb{Q} \iff \sigma(\alpha) = \alpha$ for all $\sigma \in \text{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$. Moreover, $\text{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^*$. In fact, let $\zeta$ be a generator of the cyclic group $\mu_N$. For example, we could take $\zeta = e^{2\pi i / N}$. Then for all $\sigma \in \text{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$, $\sigma(\zeta)$ is another generator of $\mu_N$, hence $\sigma(\zeta) = \zeta^a$ for some integer $a$ mod $N$, necessarily relatively prime to $N$. Viewing $\alpha$ as an element of $(\mathbb{Z}/N\mathbb{Z})^*$, the map $\sigma \mapsto a$ sets up an isomorphism $\text{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^*$. Note also that, if $\sigma(\zeta) = \zeta^a$, then $\sigma(\lambda) = \lambda^a$ for all $\lambda \in \mu_N$. 

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If $V$ is a $G$-representation, then $\chi_V(g) = \sum_i \lambda_i$, where the $\lambda_i \in \mathbb{Q}(\mu_N)$. Moreover, given $\sigma \in \text{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ corresponding to $a \in (\mathbb{Z}/N\mathbb{Z})^*$,

$$\sigma(\chi_V(g)) = \sum_i \sigma(\lambda_i) = \sum_i \lambda_i^a = \chi_V(g^a).$$

Thus $\chi_V(g) \in \mathbb{Q}$ for every representation $V$ of $G$ and every element $g \in G$ if and only if $\chi_V(g) = \chi_V(g^a)$ for every representation $V$ of $G$, every element $g \in G$, and every $a \in \mathbb{Z}$ which is relatively prime to $N$. Since the functions $\chi_V$ span the space of class functions, this is the case if and only if $g$ is conjugate to $g^a$ for every $g \in G$ and every $a \in \mathbb{Z}$ which is relatively prime to $N$. Finally, since $N$ and $\#(G)$ have the same prime factors, $a$ is relatively prime to $N$ if and only if $a$ is relatively prime to $\#(G)$.

**Example 6.3.** (1) The symmetric group $S_n$ has the property that, for every $\sigma \in S_n$ and every $a \in \mathbb{Z}$ which is relatively prime to $\#(S_n) = n!$, $\sigma^a$ is conjugate to $\sigma$. In fact, we have seen that $\sigma = \gamma_1 \cdots \gamma_\ell$, where the $\gamma_i$ are pairwise disjoint cycles of lengths $n_i > 1$. Since the $\gamma_i$ commute,

$$\sigma^a = (\gamma_1 \cdots \gamma_\ell)^a = \gamma_1^a \cdots \gamma_\ell^a.$$  

As each $\gamma_i$ is a cycle of length $n_i$ and $\gcd(a, n_i) = 1$ since $n_i$ divides $n!$, a Modern Algebra I argument shows that $\gamma_i^a$ is an $n_i$-cycle for every $i$. Also, the elements of $\{1, \ldots, n\}$ appearing in $\gamma_i^a$ are the same as the elements appearing in $\gamma_i$, so that $\gamma_1^a, \ldots, \gamma_\ell^a$ are disjoint cycles of lengths $n_i$. As we have seen before, this implies that $\sigma^a = \gamma_1^a \cdots \gamma_\ell^a$ is conjugate to $\sigma$. We shall see that, in fact, every representation of $S_n$ can be defined over $\mathbb{Q}$. 

(2) For the quaternion group $Q$, as $\#(Q) = 8$, $a \in \mathbb{Z}$ is relatively prime to $\#(Q)$ if and only if $a$ is odd. For odd $a$, it is easy to check that $g^a$ is conjugate to $g$ for every $g \in Q$. For example, $1^a = 1$, $(-1)^a = -1$, and $(\pm i)^a = \pm i$ if $a \equiv 1$ (mod 4) and $(\pm i)^a = \mp i$ if $a \equiv 3$ (mod 4). But $i$ and $-i$ are conjugate, since e.g., $-i = ji j^{-1}$. Thus (as is easy to see directly) $\chi_V(g) \in \mathbb{Q}$ for every representation $V$ of $Q$ and every element $g \in Q$. On the other hand, the irreducible 2-dimensional representation of $Q$ cannot be defined over $\mathbb{Q}$. In fact, we have seen that it cannot be defined over $\mathbb{R}$.