1 Conjugacy classes and Young diagrams

Let us recall what we know about irreducible representations of $S_n$ so far: we have the two 1-dimensional representations $\mathbb{C}$ and $\mathbb{C}(\varepsilon)$, and an irreducible representation $V$ of dimension $n - 1$ which satisfies: $V \oplus \mathbb{C} \cong \mathbb{C}[S_n/S_{n-1}]$, where $\mathbb{C}[S_n/S_{n-1}]$ is the standard permutation representation of $S_n$ coming from its action on $\{1, \ldots, n\}$. There is also the irreducible representation $V \otimes \varepsilon$, which is not isomorphic to $V$ once $n \geq 4$. Our goal in this set of notes will be to describe a construction of all of the irreducible representations of $S_n$.

We begin by recalling the usual description of the conjugacy classes in $S_n$. Every $\sigma \in S_n$ can be written as $\gamma_1 \cdots \gamma_k$, where the $\gamma_i$ are pairwise disjoint $n_i$-cycles and the product is unique up to order. Here the identity 1 corresponds to the empty product ($k = 0$). We may as well reorder so that $n_1 \geq n_2 \geq \cdots \geq n_k$, so that the $n_i$ form a non-increasing sequence of integers at least 2 and $\sum_{i=1}^k n_i \leq n$. We will refer to the sequence $(n_1, \ldots, n_k)$ as the cycle type of $\sigma$. For example, an $r$-cycle has cycle type $r$. The element $(1, 2)(3, 4, 5)$ has cycle type $(3, 2)$. Two elements of $S_n$ are conjugate $\iff$ they have the same cycle type.

It is convenient to rewrite this description of the conjugacy classes via partitions:

Definition 1.1. A partition $\lambda$ of $n$, which we write symbolically as $\lambda \vdash n$, is a weakly decreasing (i.e. non-increasing) sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$, such that $\sum_{i=1}^\ell \lambda_i = n$.

Note that, given a cycle type, i.e. a non-increasing sequence of integers $n_1 \geq n_2 \geq \cdots \geq n_k$ at least 2 and $\sum_{i=1}^\ell n_i \leq n$, we can always enlarge the sequence to a partition by considering $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1 \geq 1 \geq \cdots \geq 1$. 

1
where the number of terms equal to 1 that we add is \( n - \sum_{i=1}^{k} n_i \). Conversely, given a partition \( \lambda \), we obtain a cycle type by dropping off all of the terms at the end with \( \lambda_i = 1 \). Thus we see that the conjugacy classes of \( S_n \) are indexed by partitions of \( n \). It is therefore reasonable to hope that the irreducible representations of \( S_n \) are also indexed by partitions.

**Definition 1.2.** Given a partition \( \lambda \vdash n \), the Young subgroup \( S_\lambda \leq S_n \) is the subgroup of \( S_n \) defined by:

\[
\sigma \in S_\lambda \iff \sigma \left( \left\{ 1, \ldots, \lambda_1 \right\} \right) = \left\{ 1, \ldots, \lambda_1 \right\}, \sigma \left( \left\{ \lambda_1 + 1, \ldots, \lambda_1 + \lambda_2 \right\} \right) = \left\{ \lambda_1 + 1, \ldots, \lambda_1 + \lambda_2 \right\}, \ldots, \text{and more generally, for all } i, 1 \leq i \leq \ell,
\]

\[
\sigma \left( \left\{ \sum_{j=1}^{i-1} \lambda_j + 1, \ldots, \sum_{j=1}^{i} \lambda_j \right\} \right) = \left\{ \sum_{j=1}^{i-1} \lambda_j + 1, \ldots, \sum_{j=1}^{i} \lambda_j \right\}.
\]

In other words, \( S_\lambda \) is the subgroup which preserves the first set of \( \lambda_1 \) consecutive elements of \( \left\{ 1, \ldots, n \right\} \), then the next set of \( \lambda_2 \) consecutive elements, and so on. Thus clearly

\[ S_\lambda \cong S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}. \]

Hence \( \#(S_\lambda) = (\lambda_1)! \cdots (\lambda_\ell)! \) and \( \#(S_n/S_\lambda) = n!/(\lambda_1)! \cdots (\lambda_\ell)! \).

For example, if \( \lambda = (n) \), then \( S_\lambda = S_n \), whereas if \( \lambda = (1, 1, \ldots, 1) \), then \( S_\lambda = \{1\} \).

Let \( M_\lambda = \mathbb{C}[S_n/S_\lambda] = \text{Ind}_{S_\lambda}^{S_n} \mathbb{C} \). The basic idea will be to locate an irreducible subspace \( S_\lambda^\lambda \) of \( M_\lambda^\lambda \) satisfying certain properties. The representations \( S_\lambda^\lambda \) will exactly be the irreducible representations of \( S_n \) up to isomorphism.

## 2 Young diagrams and Young tableaux

**Definition 2.1.** Given a partition \( \lambda \vdash n \), its Young diagram is given by drawing \( n \) boxes in \( \ell \) rows, flush left, with the \( i \)th row having \( \lambda_i \) boxes.

For example, given \( \lambda = (3, 2, 1, 1, 1) \vdash 8 \) its Young diagram is

```
1 1 1 2 2 3
```

At the two extremes, for \( \lambda = (n) \), the corresponding diagram is

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and for \( \lambda = (1,1,\ldots,1) \), the corresponding diagram is

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We define an operation of transpose (written \( \lambda \mapsto \lambda^T \)) on Young diagrams by switching rows and columns. For example, with \( \lambda = (3,2,1,1,1) \vdash 8 \) as before, the transpose diagram is

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which corresponds to \( \lambda^T = (5,2,1) \). Likewise \( (n)^T = (1,1,\ldots,1) \). Clearly \( (\lambda^T)^T = \lambda \).

Next, we define a partial order on the set of all partitions:

**Definition 2.2.** Suppose that \( \lambda, \mu \vdash n \), where \( \lambda = (\lambda_1,\ldots,\lambda_\ell) \) and \( \mu = (\mu_1,\ldots,\mu_m) \). Then \( \lambda \) dominates \( \mu \), written \( \lambda \trianglerighteq \mu \), if, for all \( i \),

\[
\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i.
\]

Here, if \( i > \ell \), we set \( \lambda_i = 0 \), and similarly if \( i > m \) we set \( \mu_i = 0 \). The definition amounts to saying that, for every \( i \), the first \( i \) rows of the Young diagram for \( \lambda \) contain at least as many boxes as the first \( i \) rows of the Young diagram for \( \mu \).

The relation \( \trianglerighteq \) is only a partial order because not every two partitions are comparable. For example, \( (5,2,1) \trianglerighteq (3,4,1) \), but \( (5,1,1,1) \) and \( (3,4,1) \) are not comparable. For every partition \( \lambda \), \( (n) \trianglerighteq \lambda \) and \( \lambda \trianglerighteq (1,1,\ldots,1) \).

The following lemma makes precise the sense in which \( \trianglerighteq \) is a partial order:
Lemma 2.3. With $\geq$ defined as above, and for all $\lambda, \mu, \nu \vdash n$,

(i) $\lambda \geq \lambda$.

(ii) If $\lambda \geq \mu$ and $\mu \geq \nu$, then $\lambda \geq \nu$.

(iii) If $\lambda \geq \mu$ and $\mu \geq \lambda$, then $\lambda = \mu$.

Proof. (i) and (ii) follow easily from the definition. As for (iii), note that by definition $\lambda_1 \geq \mu_1$ and $\mu_1 \geq \lambda_1$, hence $\lambda_1 = \mu_1$. Assume inductively that we have shown that $\lambda_k = \mu_k$, $k \leq i - 1$. Then since $\lambda \geq \mu$,

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i,$$

and hence $\lambda_i \geq \mu_i$. By symmetry, $\mu_i \geq \lambda_i$. Hence $\lambda_i = \mu_i$, completing the inductive step and hence the proof of (iii).

Definition 2.4. Given $\lambda \vdash n$, a $\lambda$-tableau $t$ or a tableau of type $\lambda$ is a labeling of the $n$ boxes of the Young diagram of $\lambda$ by the elements of $\{1, \ldots, n\}$, in other words a way to fill in the $n$ boxes of the Young diagram with the elements of $\{1, \ldots, n\}$, using each element exactly once. Hence, given $\lambda$, there are exactly $n!$ tableau of type $\lambda$. For example, given $\lambda$, the basic $\lambda$-tableau $t_0$ is obtained by filling in the boxes consecutively: for $\lambda = (3, 2, 1) \vdash 6$, the basic tableau $t_0$ is

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 \\
6 \\
\end{array}
\]

Two $\lambda$-tableaux $t_1$ and $t_2$ are equivalent, written $t_1 \sim t_2$, if, for every $i$, the set of entries in the $i^{th}$ row of $t_1$ is the same as the set of entries in the $i^{th}$ row of $t_2$. In other words, $t_2$ is obtained from $t_1$ by permuting each row of $t_1$. For example,

\[
\begin{array}{ccc}
3 & 2 & 6 \\
4 & 1 \\
5 \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
6 & 2 & 3 \\
1 & 4 \\
5 \\
\end{array}
\]
are equivalent.

We write the equivalence class containing \( t \) as \( [t] \). An equivalence class of \( \lambda \)-tableaux is called a \( \lambda \)-tabloid or a tabloid of type \( \lambda \).

Clearly, \( S_n \) acts transitively on the set of \( \lambda \)-tableaux and preserves the equivalence relation \( \sim \). Thus \( S_n \) acts transitively on the set of \( \lambda \)-tabloids. If \( t_0 \) is the basic \( \lambda \)-tableau, then the isotropy subgroup of \( t_0 \) is \( S_\lambda \), the Young subgroup. Hence we can identify the set of all \( \lambda \)-tabloids with \( S_n/S_\lambda \). In particular, there are \( n!/\lambda_1! \cdots \lambda_t! \) \( \lambda \)-tabloids.

Given a \( \lambda \)-tableau \( t \), we can define a \( \lambda^T \)-tableau \( t^T \) in the obvious way. Clearly, if \( \sigma \in S_n \), then \( (\sigma \cdot t)^T = \sigma \cdot (t^T) \). However, if \( t_1 \sim t_2 \), then \( t_1^T \) and \( t_2^T \) are not in general equivalent.

As before, let \( M_\lambda = \mathbb{C}[S_n/S_\lambda] = \text{Ind}_{S_\lambda}^{S_n} \mathbb{C} \). We view \( M_\lambda \) as having a basis consisting of \( \lambda \)-tabloids. Our goal will be to find an irreducible subspace \( S_\lambda \) of \( M_\lambda \) with the property that \( \dim \text{Hom}_{S_n}(S_\lambda, M_\mu) = 1 \), i.e., that the multiplicity of \( S_\lambda \) in \( M_\mu \) is 1, and such that \( \text{Hom}_{S_n}(S_\lambda, M_\mu) \neq 0 \Rightarrow \lambda \triangleright \mu \).

**Example 2.5.**

1. If \( \lambda = (n) \), then \( S_\lambda = S_n \), \( M^{(n)} \) is the trivial representation \( \mathbb{C} \), and necessarily \( S^{(n)} = \mathbb{C} \). Note that \( (n) \triangleright \mu \) for every partition \( \mu \vdash n \), and also that the trivial representation occurs in \( M_\mu \) for every \( \mu \) because \( M_\mu \) is a permutation representation.

2. If \( \lambda = (1, \ldots, 1) \), then \( M^{(1,\ldots,1)} \) is the regular representation. We will see that \( S^{(1,\ldots,1)} = \mathbb{C}(\varepsilon) \). Note that \( \mu \triangleright (1, \ldots, 1) \) for every \( \mu \vdash n \). Likewise, \( \text{Hom}_{S_n}(S_\mu, M^{(1,\ldots,1)}) \neq 0 \) since every irreducible representation is isomorphic to a subspace of the regular representation.

3. If \( \lambda = (n-1, 1) \), then \( S_\lambda \cong S_{n-1} \) and \( M^{(n-1,1)} \) is the permutation representation of \( S_n \) acting on \( \{1, \ldots, n\} \). Hence \( M^{(n-1,1)} \cong \mathbb{C} \oplus V \), where \( V \) is irreducible of dimension \( n-1 \). Correspondingly, if \( \lambda \triangleright (n-1, 1) \), then either \( \lambda = (n) \) or \( \lambda = (n-1, 1) \).

### 3 Row and column stabilizers; polytabloids

**Definition 3.1.** Let \( t \) be a \( \lambda \)-tableau with associated tabloid \( [t] \). We define the row stabilizer \( R_t \) to be the subgroup of \( S_n \) consisting of all elements \( \sigma \) such that, for every \( i \), \( \sigma \) preserves the set of elements in the \( i \)-th row of \( t \). Equivalently, \( R_t \) is the isotropy subgroup of the associated tabloid \( [t] \). For \( t = t_0 \), \( R_{t_0} = S_\lambda \) is the Young subgroup. For a general \( t \), \( R_t \) is conjugate to \( S_\lambda \) as we shall see shortly.
We likewise define the column stabilizer $C_t$ to be the subgroup of $S_n$ consisting of all elements $\sigma$ such that, for every $i$, $\sigma$ preserves the set of elements in the $i^{th}$ column of $t$. Thus $C_t$ is the isotropy subgroup of $[t^T]$, so that $C_t = R_t^T$, and hence $C_t$ is conjugate to $S_{\lambda^T}$. However, $C_t$ depends on the tableau $t$, not just on the tabloid $[t]$.

**Lemma 3.2.** For all tableaux $t$ and all $\sigma \in S_n$,

(i) $R_t \cap C_t = \{1\}$.

(ii) $R_{\sigma \cdot t} = \sigma R_t \sigma^{-1}$ and $C_{\sigma \cdot t} = \sigma C_t \sigma^{-1}$.

**Proof.** (i) Let $a$ be the $(i,j)^{th}$ entry of $t$, i.e. $a$ lies in the $i^{th}$ row and $j^{th}$ column of $t$. If $\sigma \in R_t \cap C_t$, then $\sigma(a)$ is also in the $i^{th}$ row and $j^{th}$ column of $t$. Thus $\sigma(a) = a$ for all $a \in \{1,\ldots,n\}$, so that $\sigma = 1$.

(ii) This is a general fact about isotropy subgroups for group actions. \(\square\)

If $t$ is a $\lambda$-tableau, we define the following element of the group algebra:

$$A_t = \sum_{\sigma \in C_t} \varepsilon(\sigma)\sigma \in \mathbb{C}[S_n].$$

Note that we sum over the column stabilizer $C_t$. Since the group algebra acts on all representations, given an $S_n$-representation $V$, we can view $A_t$ as defining a linear map $V \rightarrow V$. In particular, $A_t$ defines a linear map $M^\mu \rightarrow M^\mu$ for all $\mu \vdash n$.

**Definition 3.3.** Given a $\lambda$-tableau $t$, the polytabloid $e_t$ associated to $t$ is the element

$$e_t = A_t([t]) = \sum_{\sigma \in C_t} \varepsilon(\sigma)\sigma \cdot [t] = \sum_{\sigma \in C_t} \varepsilon(\sigma)[\sigma \cdot t] \in M^\lambda.$$

**Remark 3.4.** As we shall see in numerous examples, $e_t$ depends on $t$, not just on $[t]$, because $C_t$ depends on $t$ and not just on $[t]$.

**Lemma 3.5.** For all tableaux $t$, $e_t \neq 0$.

**Proof.** Note first that, if $\sigma \in C_t$ and $\sigma \cdot [t] = [t]$, then $\sigma \in R_t$ and hence $\sigma \in R_t \cap C_t = \{1\}$. Likewise, if $\sigma_1, \sigma_2 \in C_t$ and $\sigma_1 \cdot [t] = \sigma_2 \cdot [t]$, then $\sigma_2^{-1}\sigma_1 = 1$ and hence $\sigma_1 = \sigma_2$. It follows that $e_t = \sum_{\sigma \in C_t} \varepsilon(\sigma)\sigma \cdot [t]$ is a sum of different basis vectors in $M^\lambda$, with coefficients $\pm 1$, and hence $e_t \neq 0$. \(\square\)

**Lemma 3.6.** For all tableaux $t$ and all $\sigma \in S_n$,
(i) $\sigma \cdot A_t = A_{\sigma \cdot t}$ as elements of the group algebra.

(ii) $\sigma \cdot e_t = e_{\sigma \cdot t}$.

Proof. (i) We have $\sigma A_t = \sum_{\tau \in C_{\sigma \cdot t}} \varepsilon(\tau) \tau$. On the other hand,

$$A_{\sigma \cdot t} = \sum_{\tau \in C_{\sigma \cdot t}} \varepsilon(\tau) \tau = \sum_{\tau \in \sigma C_t} \varepsilon(\tau) \tau \sigma = \sum_{\tau \in \sigma C_t} \varepsilon(\tau) \sigma \tau.$$ 

Comparing, we see that $A_{\sigma \cdot t} = A_{\sigma \cdot t}$.

(ii) By definition,

$$\sigma \cdot e_t = \sigma A_t([t]) = A_{\sigma \cdot t}([t]) = A_{\sigma \cdot t}([\sigma \cdot t]) = e_{\sigma \cdot t}.$$ 

We now define the representation $S^\lambda$. The idea is as follows: let $G$ be a finite group and $V$ an irreducible $G$-representation. For a fixed vector $v \in V$, the span of the set

$$G \cdot v = \{ \rho_V(g)(v) : g \in G \}$$

is clearly a nonzero $G$-invariant subspace of $V$. If moreover $v \in W$, where $W$ is an irreducible subspace of $V$, then this span is a nonzero $G$-invariant subspace of $W$, hence must equal $W$.

Definition 3.7. Given $\lambda \vdash n$, define $S^\lambda$, the Specht representation, to be the span of the polytabloids $e_t$, where $t$ is a $\lambda$-tableau. Since $\sigma \cdot e_t = e_{\sigma \cdot t}$, $S^\lambda$ is an $S_n$-invariant subspace of $M^\lambda$, hence an $S_n$-representation.

Example 3.8. (1) If $\lambda = (n)$, then $M^{(n)} = \mathbb{C}$ with the trivial action of $S_n$. Here, there is only one tableau $[t]$, $C_t = \{1\}$, and $A_t = \text{Id}$.

(2) Let $\lambda = (n - 1, 1)$. Then every tableau $t$ is of the form

$$\begin{array}{ccc}
* & * & * & \ldots & * \\
\ast & & & & \\
\end{array}$$

$|k$
for a unique \( k = k(t), \ 1 \leq k \leq n \). Moreover, two tableaux \( t_1 \) and \( t_2 \) are equivalent \( \iff k(t_1) = k(t_2) \). Hence the \((n - 1,1)\)-tabloids are indexed by \( k \in \{1, \ldots, n\} \). Let \([k]\) denote the corresponding equivalence class. Clearly \( \sigma \cdot [k] = [\sigma(k)] \). Thus \( M^{(n-1,1)} \cong \mathbb{C}^n \), with basis vectors \([1], \ldots, [n]\), and the \( S_n \)-action is the same as the standard permutation representation. If \( t \in [k] \), let the entry in the first row and column of \( t \) be \( i \), so that \( t \) is of the form

\[
\begin{array}{cccc}
  i & * & * & \cdots & * \\
  k &
\end{array}
\]

Then \( C_t = \{1, (ik)\} \). Note that \( C_t \) depends on \( t \), not just \([t] = [k]\). Hence \( A_t([t]) = [k] + \varepsilon((ik))(ik) \cdot [k] = [k] - [i] \). The vectors \([k] - [i]\) are not linearly independent, and their span in \( M^{(n-1,1)} \cong \mathbb{C}^n \) is easily seen to be

\[
\left\{ a_1[1] + \cdots + a_n[n] : \sum_{i=1}^n a_i = 0 \right\}.
\]

Thus \( S^{(n-1,1)} \cong V \), the standard irreducible representation of dimension \( n - 1 \) of \( S_n \).

(3) For \( \lambda = (1,1,\ldots,1) \), \( M^{(1,1,\ldots,1)} = \mathbb{C}[S_n] \), \( C_t = S_n \) for every \( t \), and \( R_t = \{1\} \). The tableaux are the same as the tabloids, and correspond to elements \( \sigma \in S_n \) via: \( t_\sigma \) is the \((1,1,\ldots,1)\)-tableau whose entries going vertically are \( \sigma(1), \sigma(2), \ldots, \sigma(n) \). Thus \( t_1 = t_0 \) is the basic tableau and \( t_\sigma = \sigma \cdot t_1 \). Then \( A_t = \sum_{\sigma \in S_n} \varepsilon(\sigma)\sigma \) for every \( t \), and

\[
e_{t_1} = A_t([t_1]) = \sum_{\sigma \in S_n} \varepsilon(\sigma)[t_\sigma].
\]

By a standard calculation in the group algebra, for every \( \tau \in S_n \),

\[
\tau \cdot A_t = A_t \cdot \tau = \varepsilon(\tau)A_t.
\]

Thus \( \tau \cdot e_{t_1} = e_{t_\tau} = A_t\tau([t_1]) = \varepsilon(\tau)e_{t_1} = \pm e_{t_1} \), so \( S^{(1,1,\ldots,1)} \) is 1-dimensional, and \( \tau(e_{t_1}) = \varepsilon(\tau)e_{t_1} \), so that \( S^{(1,1,\ldots,1)} \cong \mathbb{C}(\varepsilon) \).

### 4 Proof of irreducibility

We begin with the following lemma:
Lemma 4.1 (Dominance lemma). Let \( \lambda, \mu \vdash n \), let \( t \) be a \( \lambda \)-tableau and let \( s \) be a \( \mu \)-tableau. Suppose that, for every \( i \), if \( a \neq b \) are two entries in the \( i \)th row of \( s \), then \( a \) and \( b \) lie in different columns of \( t \). Then \( \lambda \succeq \mu \).

Proof. We first establish a claim which we shall also use:

Claim 4.2. With hypotheses as above, there exists a \( \sigma \in C_t \) such that, after replacing \( t \) by \( \sigma \cdot t \), for every \( i \), the elements in the first \( i \) rows of \( s \) all appear in the first \( i \) rows of \( t \). Equivalently, if \( S_i \) is the set of elements in the \( i \)th row of \( s \) and \( T_j \) is the set of elements in the \( j \)th row of \( t \), then, for every \( i \), \( \sigma(S_i) \subseteq \bigcup_{j \leq i} T_j \).

First let us show that the claim implies the lemma. Assuming the claim, for every \( i \) there are \( \mu_1 + \cdots + \mu_i \) elements in the first \( i \) rows of \( s \). Since they all appear in the first \( i \) rows of \( t \), the number of elements in the first \( i \) rows of \( t \), namely \( \lambda_1 + \cdots + \lambda_i \), has to be at least as large as \( \mu_1 + \cdots + \mu_i \). In other words, for every \( i \),

\[
\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i,
\]

and hence \( \lambda \succeq \mu \).

Proof of the claim. Note that the hypotheses of the lemma are unchanged by applying column permutations to \( t \), i.e. by replacing \( t \) by \( \sigma \cdot t \) for \( \sigma \in C_t \). We will give an inductive construct of an appropriate \( \sigma \).

For \( i = 1 \), the entries of the first row of \( s \) are in different columns of \( t \). In particular there are \( \lambda_1 \geq \mu_1 \) columns of \( t \). Permute each column of \( t \) containing an element of the first row of \( s \) by moving the given element into the first row (for example, by a transposition if it is not already in the first row). This replaces \( t \) by \( \sigma_1 \cdot t \) for some \( \sigma_1 \in C_t \).

For the inductive step of the construction, suppose that we have found a \( \sigma_i \in C_t \) such that, after replacing \( t \) by \( \sigma_i \cdot t \), the elements in the first \( i \) rows of \( s \) all appear in the first \( i \) rows of \( t \). Now consider the entries in the \( (i + 1) \)st row of \( s \). If any of these entries appear in one of the first \( (i + 1) \)st rows of \( t \), we leave the corresponding columns alone. If some entry \( a \) appears in the \( j \)th row of \( t \) with \( j > i + 1 \), suppose that \( a \) is also in the \( k \)th column of \( t \). Then no other entry in the \( (i + 1) \)st row of \( s \) lies in the \( k \)th column of \( t \). Also, since \( a \) lies below the \( (i + 1) \)st row of \( t \), the \( k \)th column of \( t \) has a nonempty intersection with \( (i + 1) \)st row of \( t \). Then we can permute the \( k \)th column of \( t \) by switching the in the \( j \)th row, namely \( a \), with the entry in the \( (i + 1) \)st row. This procedure doesn’t affect the first \( i \) rows, and can be done independently for each entry in the \( (i + 1) \)st row of \( s \) which lies in in
the $j^{th}$ row of $t$ for some $j > i + 1$, since these all lie in different columns. We thus modify $\sigma_i \cdot t$ by a column permutation, and hence $t$ by a column permutation $\sigma_{i+1}$, so that $\sigma_{i+1} \cdot t$ has the desired properties. This completes the inductive step of the construction.

Recall that $M^\lambda$ has a basis consisting of the $\lambda$-tabloids $[t]$. We can introduce a positive definite Hermitian inner product $\langle \cdot, \cdot \rangle$ on $M^\lambda$ by decreeing that this basis is unitary, i.e. that

$$\langle [t_1], [t_2] \rangle = \begin{cases} 1, & \text{if } [t_1] = [t_2]; \\ 0, & \text{otherwise.} \end{cases}$$

Since $S_n$ acts on $M^\lambda$ by permuting the basis vectors, this Hermitian inner product is $S_n$-invariant. In what follows, when we speak about orthogonality, it will be with respect to $\langle \cdot, \cdot \rangle$.

The key technical result we need now is:

**Theorem 4.3** (Submodule theorem). Let $V$ be an $S_n$-invariant subspace of $M^\lambda$. Then either $S^\lambda \subseteq V$ or $V \subseteq (S^\lambda)^\perp$.

**Proof.** We start with the following lemma:

**Lemma 4.4.** Suppose that $\lambda, \mu \vdash n$, that $t$ is a $\lambda$-tableau and that $s$ is a $\mu$-tableau. If $A_t([s]) \neq 0$, then $\lambda \trianglerighteq \mu$. If moreover $\lambda = \mu$, then $A_t([s]) = \pm e_t$.

**Proof.** First, we claim that, if $A_t([s]) \neq 0$, then for every $i$, if $a$ and $b$ are two elements in the $i^{th}$ row of $s$, then $a$ and $b$ are in different columns of $t$. The dominance lemma then implies that $\lambda \trianglerighteq \mu$. To see the claim, suppose by contradiction that $a$ and $b$ are in the same column of $t$. Then $(ab) \in C_t$, and also $(ab) \in R_s$, so that $(ab) \cdot [s] = s$. Let $H = \{1, (ab)\} \leq C_t$ be the subgroup generated by $(ab)$. We can then break $C_t$ up into the left cosets for $H$: if $\sigma_1, \ldots, \sigma_N$ are a set of representatives for $C_t/H$, then

$$C_t = \{\sigma_1, \sigma_1 \cdot (ab), \ldots, \sigma_N, \sigma_N \cdot (ab)\}.$$

Then, writing the elements of $C_t$ as above,

$$A_t([s]) = \sum_{\sigma \in C_t} \varepsilon(\sigma)\sigma \cdot [s] = \sum_{i=1}^N (\varepsilon(\sigma_i)\sigma_i \cdot [s] + \varepsilon(\sigma_i \cdot (ab))\sigma_i \cdot (ab) \cdot [s])$$

$$= \sum_{i=1}^N (\varepsilon(\sigma_i)\sigma_i \cdot [s] - \varepsilon(\sigma_i)\sigma_i \cdot [s]) = 0,$$
where we have used the fact that \((ab) \cdot [s] = [s]\) and that 
\[
\varepsilon(\sigma_i \cdot (ab)) = \varepsilon(\sigma_i)\varepsilon((ab)) = -\varepsilon(\sigma_i).
\]

But this contradicts the assumption that \(A_t([s]) \neq 0\).

Now suppose that \(\lambda = \mu\) and that \(A_t([s]) \neq 0\). As we have seen, the hypotheses of the dominance lemma hold. Then by Claim 4.2 there exists a set of tableaux, \(\mathcal{E}\) and every \(v \in V\) is either 0 or \(\pm e_t\). Thus, for all \(M\) is an invariant subspace of \(\lambda\)-tableaux \(\mathcal{E}\). Returning to the proof of the submodule theorem, let \(V\) be an \(S_n\)-invariant subspace of \(M^\lambda\). Then \(\mathbb{C}[S_n](V) \subseteq V\) and hence \(A_t(v) \in V\) for every \(\lambda\)-tableau \(t\) and every \(v \in V\). As \(A_t(M^\lambda) = \mathbb{C} \cdot e_t\), \(e_t \in V\) as long as there exists a \(v \in V\) such that \(A_t(v) \neq 0\). In this case, since \(\sigma \cdot e_t = e_{\sigma \cdot t}\) and \(V\) is \(S_n\)-invariant, \(e_{\sigma \cdot t} \in V\) for all \(\sigma \in S_n\). Since \(S_n\) acts transitively on the set of tableaux, \(e_s \in V\) for every tableau \(s\). As \(S^\lambda\) is generated by the \(e_s\), \(S^\lambda \subseteq V\).

Otherwise, \(A_t(v) = 0\) for every tableau \(t\) and \(v \in V\). Since the inner product \(\langle \cdot, \cdot \rangle\) is \(S_n\)-invariant, \(\langle \sigma(v), w \rangle = \langle v, \sigma^{-1}(w) \rangle\) for all \(v, w \in M^\lambda\). Then, for all \(v, w \in M^\lambda\), \(\langle A_t(v), w \rangle = \langle v, A_t^*(w) \rangle\), where 
\[
A_t^* = \sum_{\sigma \in C_t} \varepsilon(\sigma)\sigma^{-1} = \sum_{\sigma \in C_t} \varepsilon(\sigma^{-1})\sigma^{-1} = \sum_{\sigma \in C_t} \varepsilon(\sigma)\sigma = A_t.
\]
Thus $A_t(v) = 0$ for all $v \in V \iff \langle v, A_t(w) \rangle = 0$ for every $w \in M^\lambda$. Since the image of $A_t$ is $\mathbb{C} \cdot e_t$, this implies that $\langle v, e_t \rangle = 0$ for every $\lambda$-tableau $t$. Since $S^\lambda$ is the span of the $e_t$, $V \subseteq (S^\lambda)^\perp$ as claimed. \hfill $\Box$

**Corollary 4.6.** $S^\lambda$ is irreducible.

*Proof.* Note that $S^\lambda \neq \{0\}$ as $e_t \neq 0$ for every $t$. If $V$ is an $S_n$ invariant subspace of $S^\lambda$, then by the submodule theorem either $S^\lambda \subseteq V$ or $V \subseteq (S^\lambda)^\perp$. In the first case, $V = S^\lambda$ since $V \subseteq S^\lambda$ and $S^\lambda \subseteq V$. In the second case, $V \subseteq S^\lambda \cap (S^\lambda)^\perp = \{0\}$. Thus every $S_n$-invariant subspace of $S^\lambda$ is either $S^\lambda$ or $\{0\}$, so that $S^\lambda$ is irreducible. \hfill $\Box$

**Corollary 4.7.** If $\text{Hom}^S(S^\lambda, M^\mu) \neq 0$, then $\lambda \trianglerighteq \mu$. Moreover, if $\lambda = \mu$, then $\dim \text{Hom}^S(S^\lambda, M^\lambda) = 1$. Thus the multiplicity of $S^\lambda$ in $M^\lambda$ is 1.

*Proof.* If $F \neq 0$, then by Schur’s lemma $F$ is injective. Thus, for every $\lambda$-tableau $t$, $F(e_t) \neq 0$.

Since there is an $S_n$-invariant isomorphism

$$M^\lambda \cong S^\lambda \oplus (S^\lambda)^\perp,$$

we can extend $F$ to an $S_n$-morphism $\tilde{F} : M^\lambda \to M^\mu$ by setting $\tilde{F} = F$ on $S^\lambda$ and $\tilde{F} = 0$ on $(S^\lambda)^\perp$. Since $\tilde{F}$ is an $S_n$-isomorphism, it commutes with the action of $\mathbb{C}[S_n]$, so that $\tilde{F} \circ A_t = A_t \circ \tilde{F}$. But $A_t([t]) = e_t$, and hence

$$F(e_t) = \tilde{F}(e_t) = \tilde{F}(A_t([t])) = A_t(\tilde{F}([t])).$$

We can write $\tilde{F}([t])$ as a linear combination of $\mu$-tabloids $[s]$. Since $F(e_t) \neq 0$, there must exist a $\mu$-tabloid $s$ such that $A_t([s]) \neq 0$. By Lemma 4.4, $\lambda \trianglerighteq \mu$. Moreover, if $\lambda = \mu$, then $A_t([s]) = \pm e_t$, so that $F(e_t) \in S^\lambda$ for all $t$. It follows that $F$ is given by $i \circ G$, where $i : S^\lambda \to M^\lambda$ is the inclusion and $G \in \text{Hom}^S(S^\lambda, S^\lambda)$. By Schur’s lemma, $\text{Hom}^S(S^\lambda, S^\lambda) = \mathbb{C} \cdot \text{Id}$. Thus every $S_n$ morphism from $S^\lambda$ to $M^\lambda$ is multiplication by a scalar, followed by inclusion, so that $\dim \text{Hom}^S(S^\lambda, M^\lambda) = 1$. \hfill $\Box$

**Corollary 4.8.** For all $\lambda, \mu \vdash n$, $S^\lambda \cong S^\mu$ as $S_n$-representations $\iff \lambda = \mu$.

*Proof.* Trivially, if $\lambda = \mu$, then $S^\lambda \cong S^\mu$. Conversely, suppose that $S^\lambda \cong S^\mu$. Then the composition of this isomorphism with the inclusion $S^\mu \subseteq M^\mu$ gives a nonzero element of $\text{Hom}^S(S^\lambda, M^\mu) \neq 0$. The previous corollary then implies that $\lambda \trianglerighteq \mu$. By symmetry, $\mu \trianglerighteq \lambda$. Hence $\lambda = \mu$. \hfill $\Box$
5 Some concluding remarks

In this final section, we make some more remarks about the irreducible representations of $S_n$, mostly without proofs.

5.1 Rationality of the representations

As we have previously noted, if $\gcd(a, n!) = 1$ and $\sigma \in S_n$, then $\sigma^a$ and $\sigma$ are conjugate, and this implies that, for every representation $V$ of $S_n$, the value of the character $\chi_V(\sigma)$ is an integer for every $\sigma \in S_n$. In fact, a stronger statement is true:

**Theorem 5.1.** The irreducible representations $S^\lambda$ of $S_n$ are defined over $\mathbb{Q}$. Hence every representation of $S_n$ can be defined over $\mathbb{Q}$.

The main point of the proof is as follows. The representation $M^\lambda$ is defined over $\mathbb{Q}$. In fact, $M^\lambda = \mathbb{C}[S_n/S_\lambda]$, with a basis consisting of the $\lambda$-tabloids $[t]$, and we can just take the corresponding $\mathbb{Q}$-vector space $M^\lambda_\mathbb{Q} = \mathbb{Q}[S_n/S_\lambda]$, with a $\mathbb{Q}$-basis consisting of the $\lambda$-tabloids $[t]$. Note that $\sigma \in S_n$ acts by permuting the tabloids, and hence the matrix corresponding to $\sigma$ has rational entries, in fact every entry is either 0 or 1. The polytabloids $e_t$ are also elements of $M^\lambda_\mathbb{Q}$, since they are linear combinations of certain tabloids with coefficients $\pm 1$. Hence they span a vector subspace of $M^\lambda$ which is also defined over $\mathbb{Q}$.

5.2 Explicit construction of some representations

We have already seen that the trivial representation $\mathbb{C}$ is isomorphic to $S^{(n)}$, that $\mathbb{C}(\varepsilon)$ is isomorphic to $S^{(1,\ldots,1)}$, and that the standard representation $V$ is isomorphic to $S^{(n-1,1)}$. Linear algebra can construct a few of the other irreducible representations directly. One basic linear algebra construction is exterior or alternating product: given a vector space $U$, we can construct a new vector space $\wedge^k U$, which is generated by expressions of the form $v_1 \wedge \cdots \wedge v_k$ which are multilinear in the $v_i$. For any collection $v_1, \ldots, v_k$ of elements of $U$, we have the basic transformation law: for all $\sigma \in S_k$

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \varepsilon(\sigma)v_1 \wedge \cdots \wedge v_k.$$ 

If $u_1, \ldots, u_d$ is a basis for $U$, then a basis for $\wedge^k U$ is given by:

$$\{u_{i_1} \wedge \cdots \wedge u_{i_k} : i_1 < \cdots < i_k\}.$$ 

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In particular \( \dim \bigwedge^k U = \binom{d}{k} \) for \( k \leq d \), and \( \dim \bigwedge^k U = 0 \) for \( k > d \). Then one can show:

**Proposition 5.2.** For \( k \leq n - 1 \), \( \bigwedge^k V = \bigwedge^k S^{(n-1,1)} \) is an irreducible representation of \( S_n \), and it is isomorphic to \( S^{(n-k,1,\ldots,1)} \).

An explicit proof is sketched in the HW.

Another construction of representations uses the symmetric product: given a vector space \( U \), we can construct a new vector space \( \text{Sym}^k U \), which is generated by expressions of the form \( v_1 \ldots v_k \) which are multilinear in the \( v_i \). For any collection \( v_1, \ldots, v_k \) of elements of \( U \), we have the basic transformation law: for all \( \sigma \in S_k \)

\[
v_{\sigma(1)} \cdots v_{\sigma(k)} = v_1 \cdots v_k.
\]

If \( u_1, \ldots, u_d \) is a basis for \( U \), then a basis for \( \text{Sym}^k U \) is given by:

\[
\{ u_{i_1} \cdots u_{i_k} : i_1 \leq \cdots \leq i_k \}.
\]

In particular \( \dim \text{Sym}^k U = \binom{d+k-1}{k} \). It is then easy to check that, for \( k \leq n/2 \), there is an injective \( S_n \)-morphism \( M^{(n-k,k)} \to \text{Sym}^k V \). Hence \( S^{(n-k,k)} \) is isomorphic to an \( S_n \)-invariant summand of \( \text{Sym}^k V \). For \( n = 2 \), it is easy to make this more explicit:

**Proposition 5.3.** \( \text{Sym}^2 V \cong \mathbb{C} \oplus V \oplus S^{(n-2,2)} \).

In fact, one can identify the subspace \( \mathbb{C} \oplus V \) explicitly as well and so give a concrete realization of \( S^{(n-2,2)} \). Note that \( \dim S^{(n-2,2)} = \frac{n(n-3)}{2} \).

### 5.3 Conjugate partitions

For every partition \( \lambda \vdash n \), we have defined the transpose \( \lambda^T \vdash n \), and \( (\lambda^T)^T = \lambda \). Note that it is possible for \( \lambda^T = \lambda \). For example, \( (n)^T = (1, \ldots, 1) \). For the representations \( S^\lambda \), we have the following result, which generalizes \( S^{(1,\ldots,1)} = \mathbb{C}(\varepsilon) \):

**Proposition 5.4.** \( S^{\lambda^T} \cong S^\lambda \otimes \varepsilon \).
5.4 Representations of the alternating group

The alternating group $A_n$ is a subgroup of $S_n$ of index two, and so we can apply our general results about restrictions of irreducible representations to subgroups of index two:

**Proposition 5.5.** Let $\lambda \vdash n$.

(i) $\lambda = \lambda^T \iff S^\lambda \cong S^\lambda \otimes \varepsilon$. In this case,

$$\text{Res}_{A_n} S^\lambda \cong \text{Res}_{A_n}^S (S^\lambda \otimes \varepsilon) \cong W \oplus W',$$

where $W$ and $W'$ are two irreducible representations of $A_n$, with $\dim W = \dim W'$ and $W$, $W'$ are not isomorphic.

(ii) $\lambda \neq \lambda^T \iff S^\lambda$ and $S^\lambda \otimes \varepsilon$. In this case,

$$\text{Res}_{A_n} S^\lambda \cong \text{Res}_{A_n}^S (S^\lambda \otimes \varepsilon)$$

is an irreducible representation of $A_n$.

Finally, every irreducible representation of $A_n$ arises in this way.

**Example 5.6.** We consider the case $n = 5$. There are two 1-dimensional representations of $S_5$, $S^{(5)} \cong \mathbb{C}$ and $S^{(1,1,1,1,1)} \cong \mathbb{C}(\varepsilon)$. There are two 4-dimensional representations, the standard representation $V = S^{(4,1)}$ and $V \otimes \varepsilon = S^{(2,1,1,1)}$, where we have used the fact that $(4,1)^T = (2,1,1,1)$ and Proposition 5.4. Next, $S^{(3,2)}$ is an irreducible representation of dimension 5, and since $(3,2)^T = (2,2,1)$, we have $S^{(2,2,1)} \cong S^{(3,2)} \otimes \varepsilon$, also of dimension 5. Finally, $\Lambda^2 V \cong S^{(3,1,1,1)}$ is irreducible of dimension 6. Since $(3,1,1)^T = (3,1,1), \Lambda^2 V \cong \Lambda^2 V \otimes \varepsilon$, and this is the only irreducible representation up to isomorphism with this property.

As a check, we add up the sums of the squares of the irreducible representations constructed above:

$$1^2 + 1^2 + 4^2 + 4^2 + 5^2 + 5^2 + 6^2 = 120 = \#(S_5),$$

as expected.

We turn now to $A_5$. The representations $\mathbb{C}$ and $\mathbb{C}(\varepsilon)$ both restrict to the trivial representation of $A_5$. The representations $V$ and $V \otimes \varepsilon$ both restrict to an irreducible representation of dimension 4, the restriction of the standard irreducible representation $V$ to $A_4$. The representations $S^{(3,2)}$ and $S^{(2,2,1)} \cong S^{(3,2)} \otimes \varepsilon$ both restrict to an irreducible representation of dimension 5. Finally, the 6-dimensional representation $\Lambda^2 V \cong S^{(3,1,1)}$ restricts
on $A_5$ to $W \oplus W'$, where $W$ and $W'$ are two non-isomorphic irreducible representations of $A_5$. Finally, every irreducible representation of $A_5$ is one of these. As a check,

$$1^2 + 4^2 + 5^2 + 3^2 + 3^2 = 60 = \#(A_5).$$

With a little more effort, we can work out the character table for $A_5$. There are 5 conjugacy classes: all 3-cycles and products of two disjoint 2-cycles are conjugate in $A_5$, but there are two different conjugacy classes of 5-cycles (any two 5-cycles are conjugate in $S_5$, but not necessarily in $A_5$).

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>$C((1, 2, 3))$</th>
<th>$C((12)(34))$</th>
<th>$C((12345))$</th>
<th>$C((21345))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_C$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_V$</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\chi_{S_{3.2}}$</td>
<td>5</td>
<td>$-1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_W$</td>
<td>3</td>
<td>0</td>
<td>$-1$</td>
<td>$\frac{1+\sqrt{5}}{2}$</td>
<td>$\frac{1-\sqrt{5}}{2}$</td>
</tr>
<tr>
<td>$\chi_{W'}$</td>
<td>3</td>
<td>0</td>
<td>$-1$</td>
<td>$\frac{1-\sqrt{5}}{2}$</td>
<td>$\frac{1+\sqrt{5}}{2}$</td>
</tr>
</tbody>
</table>

Note: The images of the 3-dimensional representations $W$ and $W'$ can be realized as a subgroup of $SO(3)$, the icosahedral group. It is the group of symmetries of a regular dodecahedron, or equivalently of a regular icosahedron.

### 5.5 Further directions

There are many other questions one might ask about representations of $S_n$. Here are two:

**Branching rules:** The group $S_n$ naturally contains $S_{n-1}$ as a subgroup and in turn is naturally a subgroup of $S_{n+1}$. Given $\lambda \vdash n$ and the irreducible representation $S^\lambda$ of $S_n$, we have the corresponding representation $\text{Res}_{S_{n-1}}^{S_n} S^\lambda$ of $S_{n-1}$ as well as the representation $\text{Ind}_{S_{n-1}}^{S_{n+1}} S^\lambda$. These can both be described in terms of the Young diagram of $\lambda$.

**Multiplication rules:** Here, given $\lambda, \mu \vdash n$, the problem is to describe the the irreducible summands and their multiplicities of the representation $S^\lambda \otimes S^\mu$.

For a discussion of these and many other questions related to representations of $S_n$, we refer to the many books on $S_n$. 

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